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NUMBER 4

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THE MATHEMATICS TEACHER

VOLUME XX

APRIL, 1927

NUMBER 4

THE PLACE AND TEACHING OF CALCULUS IN SECONDARY SCHOOLS¹

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I. JUSTIFICATION FOR SUCH A COURSE

A. Curricula

The past two decades have seen many changes in the mathematics of secondary schools both in aim, method, and content. This movement to vitalize mathematics and make it more appreciative was begun in 1901 by Professor Perry who was at that time in charge of certain apprenticeship schools in London and who felt that the traditional mathematics of those schools meant little to the pupils in their later work. The reform movement in this country was started by Professor E. H. Moore of the University of Chicago. In 1903 associations of mathematics teachers were formed in various parts of the United States and mathematics magazines were established in order to spread the movement among classroom teachers. After twenty years of persistent effort on the part of reformers, we can see a marked change in aim, method, and content of mathematical curricula through new types of text books, efforts of progressive teachers, and through investigations and reports of national and international committees culminating in the Reorganization of Mathematics in Secondary Education by the National Committee of Mathematics Requirements under the auspices of the Mathematics Association of America. This noteworthy publication furnishes a national basis from which future improvement is assured.

In this movement the reformers have fixed their attention chiefly upon the junior high school period to make the mathe-

¹ M. A. Thesis, Boston University, 1925.

matics of these years more meaningful and to bridge the gap between the junior and senior high school periods through the reorganization of the mathematics of the ninth school year. Although several years are necessary to determine the most valuable content and method of organization, countless numbers of teachers have come to feel a need for breaking away from the traditional curricula and, no doubt, are generally in accord with the general aims and methods of mathematical instruction.

The reorganization of mathematics of the senior high school has taken place much more slowly because of the conservatism of secondary school teachers and because of the requirements for college entrance. Now that the College Entrance Board has abridged the requirements, more time may be given to initiative and originality in teaching. If a desire for reorganization is stimulated and only a small degree of success is apparent as yet, this sign of the times is a healthy one. However, many experimental² and public² high schools have reported excellent results in reorganized courses.

In the reorganization of senior high school mathematics the correlation has been between algebra and geometry, and the formula and the graph. The introduction of the elementary calculus must of necessity be the culmination of the high school course—a natural outgrowth of a revision of courses below. An examination of the junior high school courses as suggested by the National Committee on Mathematics Requirements will seek and reinforce a revision of senior high school mathematics. The junior high school plans are listed below from the bulletin.³

PLAN A

First year: Applications of arithmetic, particularly in such lines as relate to the home, to thrift, and to the various school subjects; intuitive geometry.

Second year: Algebra; applied arithmetic, particularly in such lines as relate to commercial, industrial, and social needs.

Third year: Algebra, trigonometry, demonstrative geometry.

By this plan the demonstrative geometry is introduced in the third year, and arithmetic is practically completed in the second.

PLAN B

First year: Applied arithmetic (as in plan A); intuitive geometry.

Second year: Algebra, intuitive geometry, trigonometry.

² See *Reorganization of Mathematics in Secondary Education*.

³ See *Reorganization of Mathematics in Secondary Education*, p. 29.

Third year: Applied arithmetic, algebra, trigonometry, demonstrative geometry.

By this plan trigonometry is taken up in two years, and the arithmetic is transferred from the second year to the third.

PLAN C

First year: Applied arithmetic, intuitive geometry, algebra.

Second year: Algebra, intuitive geometry.

Third year: Trigonometry, demonstrative geometry, applied arithmetic.
By this plan algebra is confined chiefly to the first two years.

PLAN D

First year: Applied arithmetic, intuitive geometry.

Second year: Intuitive geometry, algebra.

Third year: Algebra, trigonometry, applied arithmetic.

By this plan algebra is confined chiefly to the first two years.

PLAN E

First year: Intuitive geometry, simple formulas, elementary principles of statistics, arithmetic.

Second year: Intuitive geometry, algebra, arithmetic.

Third year: Geometry, numerical trigonometry, arithmetic.

It seems to have been the tendency in this reform movement to eliminate from the junior high school all material which has no direct bearing upon the pupil's later mathematical career, or such material that is meaningless to the child at that particular stage of his career.

The following plans for reorganizing the mathematics curricula of senior high schools are suggested by the committee. As in previous outline, no one plan is considered superior to the other.

PLAN A

First year (tenth school year): Plane demonstrative geometry, algebra.

Second year (eleventh school year): Statistics, trigonometry, solid geometry.

Third year (twelfth school year): Calculus, other elective.

PLAN B

First year: Plane demonstrative geometry, solid geometry.

Second year: Algebra, trigonometry, statistics.

Third year: Calculus, other elective.

PLAN C

First year: Plane demonstrative geometry, trigonometry.

Second year: Solid geometry, algebra, statistics.

Third year: Calculus, other elective.

PLAN D

First year: Algebra, statistics, trigonometry.

Second year: Plane geometry, solid geometry.

Third year: Calculus, other elective.

Those who have had any influence in bringing about these changes in the curricula of the junior and senior high schools feel that it is better to substitute the simpler parts of higher mathematics for the more difficult parts of arithmetic, algebra, and plane and solid geometry. The student who does not go on to college has a richer and more useful knowledge of mathematics and the one who does go on has a broader outlook upon subsequent mathematical courses.

Before justifying further the place of the elementary calculus in the senior high school, we might profit by a survey of the mathematical curricula of foreign countries that correspond to our high schools. From a careful study of authoritative reports⁴ we may draw these general conclusions:

Tenth school year (15-16 years of age approximately): Realschule of Russia teaches plane analytics and elements of the infinitesimal calculus; the Realgymnasium and Oberrealschule of Germany, analytic geometry.

Eleventh school year (16-17 years of age): Introduction to analytic geometry in the Gymnasium in Austria is given and a more difficult course in the Realschule. In the latter school easy differential and integral calculus is introduced to simplify or make more intensive the knowledge of physics. In the Oberrealschule in Germany analytic geometry appears; in the curricula of the three main school of Austria, in the Lycees of France, of the Realschule and Gymnasium of Hungary and of the Realgymnasium of Sweden.

Twelfth school year (17-18 years of age): Analytic geometry is taught in the gymnasium schools of Holland and Hungary; analytic geometry and the calculus in Austria, Belgium, Denmark, England, France, Germany, Roumania, Sweden, and Switzerland. The work in these two subjects ranges from simple introductory courses to courses equivalent to our first course in college.

From these reports of the International Committee on the Teaching of Mathematics, it seems that pupils abroad are by the end of the twelfth school year about two years ahead of our American pupils because of a more profitable arrangement of mathematics curricula in their schools. It may be contended that these foreign schools have a select group to work with but let it be remembered that any course in mathematics beyond the ninth or tenth year, of necessity, calls for a select group, viz. those who are preparing for college and others who are mathematically inclined.

⁴ International Committee on Teaching of Mathematics; "Place of Elementary Calculus in the Senior High School," by Rosenburger.

B. Teachers

In making a revision of our curricula in mathematics, we must consider the training of our teachers. Even junior high school teachers should have the minimum requirements of senior high school teachers in order to obtain a proper appreciative attitude toward this elementary content. The departments of mathematics of any reputable college, like any other departments has a definite course of study for students who are majoring in that department and in addition offers special courses for prospective teachers. The department of mathematics⁵ at Brown University, for instance, has laid down a minimum course of study for prospective teachers of mathematics who wish the backing of the department in entering their career:

Plane trigonometry	3 semester hours
Higher algebra	3
Solid geometry	3
Plane analytic geometry	4
Differential and integral calculus	8
Teacher's course in algebra	6
Teacher's course in geometry	6
<hr/>	
Total	33

Since there are more teachers to select from than formerly and since teachers nowadays are more inclined to seek improvement and advancement through attendance at summer sessions, it is very probable that there will be a sufficient supply of qualified teachers soon and no doubt there are a sufficient number now in the strong high schools. Moreover, it is needless to say that our mathematics teachers must have sufficient training in education courses also.

It might be worth while to examine the preparation of mathematics teachers of some of the stronger high schools. The following information has been culled from Chapter XII of the Reorganization of Mathematics in Secondary School Mathematics:

Cass Technical High School, Detroit, Michigan; thirteen full time mathematics teachers who average 31.8 semester hours in mathematics and 13.8 in education.

⁵ Bureau of Education Bulletin No. 7, 1917, by Raymond C. Archibald, Associate Professor of Mathematics, Brown University.

English High School, Boston; seven full time mathematics teachers, five part time, all of whom have had training beyond the master's degree.

Horace Mann School, New York City; four mathematics teachers with training beyond the master's degree.

The Rochester, New York, system of training its junior high school teachers of mathematics should be an inspiration to all progressive school systems. First of all the city has a supervisor of junior high school mathematics and then proceeds to train elementary school teachers in content and method by extension courses or Saturday institutes. These junior high school teachers have all had courses in intermediate and advanced algebra, plane and solid geometry, trigonometry, and the elements of the calculus given at the University of Rochester. A similar method of selecting senior high school mathematics teachers would surely make for improvement here, also.

C. Importance of Calculus

Now that we have made a place for the elementary calculus and believe that we have teachers available, for the larger high schools at any rate, we may be called upon to justify its place in the curriculum.

Of course, all the mathematics that the average person needs to know may be found in the first nine grades. However, higher mathematics has contributed so much of practical and cultural value to civilization that the very existence of our social fabric depends upon it. Dr. David Eugene Smith⁶ tells us that "if by some chance every trace of mathematical material were removed from the world, every mill in the whole world would slow down and every large concern would close until it could replace its accounts, its statistical material, its formulas for work, its measures, tables, and its computing machinery * * * every ship in the seven seas would be stricken with blindness. Not a rivet would be driven in a skyscraper in New York City * * * Wall Street would close its portals; the engineering world would awaken tomorrow morning to a living death; the mines would shut down, and trade would relapse to the conditions of barter as in the days of savagery."

The practical value of mathematics may be readily appreci-

⁶ *Teachers College Record*, May 1917.

ated but the cultural value depends upon the method of instruction. Its socializing value rests in the pupil's proper attitude toward the specialist upon whose work our social life depends. Even though calculus should not be of any practical value to any particular child, he should not be denied the privilege of enjoying its pursuit but should have as good a right to develop that ability as any other ability. No one would think of denying one the privilege of developing one's musical ability; then why should one who enjoys mathematics for its own sake not have the same opportunity to develop his own peculiar gift? Neither may be of any practical value or yield any financial returns, but are both not truth and beauty in different attire?

Now what advantage has the calculus over other branches of mathematics which may be taken during the senior year of the high school course? Let us consider other possibilities. There is analytic geometry but it has no applications; higher algebra including permutations and combinations, theory of equations and theory of determinants does not lead anywhere; projective geometry although very interesting has no applications. Now an introductory course in the elementary calculus would not only prove exceedingly fascinating to the student fond of mathematics but it would give him a notion of the calculus as a powerful tool on which much of our material civilization depends and would serve as an introduction to later college courses or as a basis for future study under self-instruction. Such a course would also open up the field of pure mathematics and would be readily appreciated as the connecting link between pure and applied mathematics.

II. SUBJECT MATTER AND PRESENTATION

A. Psychological vs. Logical Method

We have already noted that it is the tendency in all subjects generally to eliminate the more difficult parts of the elementary matter and substitute the easier parts of the more advanced. This arrangement is successful only when we relate the work to the child's experience. Since the mental development of the child follows in general the mental development of the race, the calculus must be presented largely in the way in which it has developed in the minds of men. It will be remembered that the new junior high school mathematics has been successful or un-

successful in proportion as the arrangement of content and method of presentation has been in keeping with the child's experience and point of view. In other words, the psychological rather than the logical method must prevail.

With this background in mind, the calculus may be developed as a continuation of algebra, geometry, and trigonometry or simultaneously with them—a unifying agency, as it were, of these three provinces.

In the first place the pupil should be taught ideas, not notations and definitions, and must be shown relationships in concrete cases before he can think in abstract terms. Gradually, then, he will form the habit of thinking of relationships between related qualities.

B. Fundamental Notions

Should we scan our mathematics curricula of the junior and senior high schools and methods of presentation, we would find that we have instilled in our students, consciously or unconsciously, the fundamental ideas of the calculus if we have had the thoroughly appreciative point of view. Besides the function concept, which unifies all mathematics, other ideas, incidentally pre-calculus, occur frequently in our regular mathematics courses. The concept of the division of a quantity indefinitely, and the idea of a variable approaching, reaching, or passing through a limit has been readily seen in many instances in demonstrative geometry. For example, inscribed and circumscribed polygons approach the circumference of the circle as a limit, if their sides are indefinitely increased; a variable chord of a circle passes through the diameter as its maximum and approaches a point at its minimum; the tangent is the limiting position of the secant; the lower and upper limits of a variable angle formed by the intersection of two tangents, a tangent and a secant, or two secants, are 0° which the variable reaches and 180° which it passes through.

Graphic methods also furnish a good basis for future work. Here the idea of dependence of one quantity upon another is constantly made use of and is readily grasped by pupils if presented in the right way. This function concept is very important in solving problems in algebra. The pupil may often see that some relationship exists but often does not realize that he

must first find a basis of relationship or starting point, so to speak, upon which the other quantities depend. If this method of attack is put into the student's hands, the difficulties of problem solving are greatly lessened and the function concept becomes more vital and meaningful.

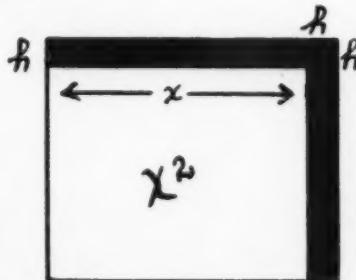
C. Calculus Methods

(a) Differentiation

1. Related Quantities.—As an introduction to calculus methods it would be well to begin with a discussion of related quantities and their graphic representation. For example, let us take the formula for the area of a square $A = x^2$. Let us show this relationship by means of a graph. This shows pictorially that for every value of x we get a definite value for A .



2. Differences.—Now suppose we have a square sheet of metal x in. on a side. Let it be heated so that it expands slightly and becomes $(x + h)$ in. on a side. In the figure below, the original area is x^2 , the increase is $2hx + h^2$. Now suppose we apply this example numerically. Let $x = 4$, $h = .006$, then $2hx + h^2 = 2 \cdot 4 \cdot .006 + .000036$.



Since the h^2 in this case is so much smaller than the other term, it may be omitted without appreciable error. We may

then call a quantity containing h a small quantity⁷ of the first order, one containing h^2 a small quantity of the second order, etc. Then if h is a very small fraction of x , second and higher orders may be omitted.

It might be well to work this example algebraically, too.

x^2 = old area; $(x+h)^2 = x^2 + 2hx + h^2$ = new area. The increase in area = new area - old = $2hx + h^2$, or omitting the second order, difference in area is $2hx = 2x$ (diff. x),⁸ where diff. x means simply the difference between the two values of x .

Let us take another example. The formula which expresses the relationship between the length of a pendulum and the time of swing is $L = 39T^2$ approximately. Find the relationship between the increase in length and the increase in the time of swing.

$$\text{New } L = 39(T+h)^2 = 39T^2 + 78Th + 39h^2.$$

$$\text{Old } L = 39T^2.$$

$$\begin{aligned}\text{Diff. } L &= 78Th + \text{second order term} \\ &= 78T(\text{Diff. } T).\end{aligned}$$

To apply this numerically, suppose a clock pendulum makes a swing in .999 sec. instead of one second, making a gain of nearly $1\frac{1}{2}$ min. a day. How much should the pendulum be lengthened? $T = .999$ and Diff. $T = 1.000 - .999$.

$$\text{Diff. } L = 78 \cdot 1 \cdot .078 \text{ which is about } 1/13 \text{ in.}$$

Then the clock will keep time if the bob is lowered $1/13$ in.

It will be noticed that the above results are not exact because of the omission of the second order term. This term is not only small, but small in comparison with the first order term.

$$\frac{\text{2d order}}{\text{1st order}} = \frac{39h^2}{78Th} = \frac{h}{2T} = \frac{.001}{.999} = \frac{1}{999},$$

which in practice is a very small fraction.

It should also be impressed upon the student that the data is obtained from measurement and is therefore at best subject to personal errors.

The students should be given other examples of similar character and should be encouraged to make up examples of their own, for in so doing the calculus ideas become more significant.

⁷ This presentation of the calculus of approximations follows Brewster's *Common Sense of the Calculus*.

⁸ Notation from Brewster's *Common Sense of the Calculus*.

Here are some suggestions:

1. What is the increase in the area of a circle when the radius is increased slightly?
2. Find the increase in the area of a right triangle when the altitude receives a small increase.
3. Find the increase in the volume of a sphere when the radius is increased slightly.

After the student has grasped these elementary ideas through experience, as it were, he will welcome more compact notations and general rules. We might introduce at this point the symbol ΔA for difference in area, Δx for difference in length of side, ΔL for Diff. L (which means the difference between the two values of the variable), etc. The pupil might compare the results of all his examples and see if he can discover any general rule which applies to all results.

From the above examples:

$$\begin{array}{ll} A = x^2, & \Delta A = 2hx \Delta x, \\ L = 39T^2, & \Delta L = 78T \Delta L, \\ A = \pi r^2, & \Delta A = 2\pi r \Delta r, \\ A = 1/2bh, & \Delta A = 1/2bh \Delta h, \\ V = 4/3r^3, & \Delta V = 4\pi r^2 \Delta r, \end{array}$$

the student is now able to write down at sight the differential (shorter name for Diff. A , etc.) since the differential of any term is found by multiplying by the index, diminishing the index by one, and multiplying by Δx .

The student should now have considerable practice in manipulation and should be encouraged to bring in original problems for which he will have to search through his previous mathematical equipment.

3. Suppose we consider another type of problem—a speed problem. Every one knows the ordinary conception of speed or rate as the distance per unit of time (hours, minutes, seconds, etc.) or algebraically speed = distance/time. Thus if a car goes 35 mi. in 2 hrs., it goes at the rate of $35/2$ or 17.5 mi. per hr. provided the speed is constant. However, this result is only an average rate and is accurate for any given interval only when the speed has been constant. For the most part, such would not be the case. One might be going at the rate of 50 mi. in one given interval and 20 mi. in another.

Example: Suppose you are in an automobile going at the rate of 45 mi. per hour, which is equivalent to 66 ft. per second. If the speedometer needle remains steady, you are going at the rate of 660 ft. in the next 10 sec., 330 ft. in 5 sec., and 66 ft. in one sec., etc. Since the needle fluctuates slowly, any of these results will be approximately true. However, the shorter the interval of time, the less likely is the needle to move perceptibly. Thus the shorter the distance, the more accurate the result.

Let S = total distance car has gone in t sec.

Then ΔS = distance it goes in Δt sec.

If Δt is taken short enough, the speed V ft. per second will be practically constant for the given time and $\Delta S = V \cdot \Delta t$ or $V = \Delta S / \Delta t$ very nearly.

Example 2. The formula for any body falling in space is given by the formula $S = 1/2gt^2$ where S is the distance fallen, t the time in seconds, g the acceleration of gravity, 32 ft./sec.

$$\begin{aligned}\text{Then } S + \Delta S &= 16(t + \bar{\Delta}t)^2 \\ &= 16(t^2 + 2t \Delta t + \bar{\Delta}t^2) \\ &S = 16t^2.\end{aligned}$$

$$\text{Subtracting, } \Delta S = 32t \Delta t + 16 \Delta t^2.$$

The smaller the interval Δt , the smaller will $(\Delta t)^2$ become, and the more closely will $32t \Delta t$ approach the ideal value. Then ΔS will equal $32t \Delta t$ or $\Delta S / \Delta t = 32t$, which gives the rate for any value of t .

Plotting this relationship between time and distance in this problem, we have the following curve:

$$\text{when } t = 1, S = 16$$

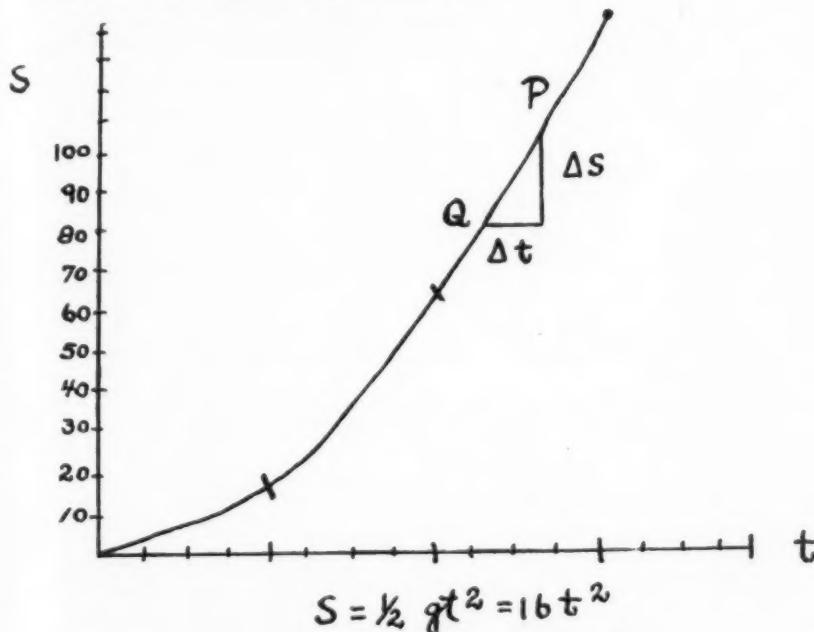
$$\text{when } t = 2, S = 64$$

$$\text{when } t = 3, S = 144$$

Now from the figure let us consider the relationship geometrically. Take any point Q on the curve and a short interval of time Δt which will give the correspondingly short distance ΔS as shown on the graph. Considering QP a straight line (which it is very approximately), we have $\Delta S / \Delta t$, which ratio is the tangent of the angle at Q , the slope or gradient of the curve at Q , or the speed or rate at this point.

4. Method of Limits.—After the student has had sufficient work in the calculus of approximations, he may be given the

general method of working out the slope of the curve. Let us plot the curve $y = x^2 + 3x + 2$. We wish to find a general method of getting the slope at any point.



Let us find the change in y as x changes slightly.

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 + 3(x + \Delta x) + 2 \\ &= x^2 + 2x\Delta x + (\Delta x)^2 + 3x + 3\Delta x + 2. \end{aligned}$$

$$\text{Subtracting, } y = x^2 + 3x + 2,$$

$$\Delta y = 2x\Delta x + (\Delta x)^2 + 3\Delta x.$$

Dividing by Δx , we have

$$\Delta y/\Delta x = 2x + 3 + \Delta x.$$

We have seen hitherto that the smaller Δx is taken, the more accurate our results. Suppose we make use of our knowledge of limits, since x is a changing quantity, and let Δx approach 0. Then Δy will also approach 0 but not at the same rate, of course, unless $x = y$ (in which case $y/x = 1$). The difficulty here will be in making the pupil see that $\Delta y/\Delta x$ approaches a finite quantity (not 0) as Δx and Δy approach 0. Give him numerical work like the following until he is satisfied that the

ratio of two infinitesimals is finite. From the equation $y = x^2$,
 $\Delta y/\Delta x = 2x + \Delta x$.

x	y	Δx	Δy	$\frac{\Delta y}{\Delta x}$
3	.09	.3	.27	$\frac{.27}{.30} = 0.900$
		.2	.16	$\frac{.16}{.20} = 0.800$
		.1	.07	$\frac{.07}{.10} = 0.700$
		.05	.0325	$\frac{.0325}{.05} = 0.650$
		.04	.0256	$\frac{.0256}{.04} = 0.640$
		.02	.0124	$\frac{.0124}{.02} = 0.62$
		.01	.0061	$\frac{.0061}{.01} = 0.610$
		.005	.003025	$\frac{.003025}{.005} = 0.602$
		.002	.001204	$\frac{.001204}{.002} = 0.602$
		.001	.000601	$\frac{.000601}{.001} = 0.601$

These show that the smaller Δx , the nearer $\Delta y/\Delta x$ comes to .6 which is the limit of the ratio $\Delta y/\Delta x$.

Going back to the previous example, let us denote the limit of this ratio $\Delta y/\Delta x$ by dy/dx . Then the limit of $\Delta y/\Delta x$ [$\lim (\Delta y/\Delta x) = 2x + 3$ for Δx in the right hand member disappears entirely since $\Delta x \rightarrow 0$].

The method may now be made more general by differentiating the general polynomial $y = ax^n + bx^{n-1} + \dots + k$ which gives $dy/dx = anx^{n-1} + b(n-1)x^{n-2} + \dots$ as the formula for differentiating an algebraic polynomial.

The student should be given further work in manipulation. Problem solving would, no doubt, be more valuable since it makes the work seem more worth while.

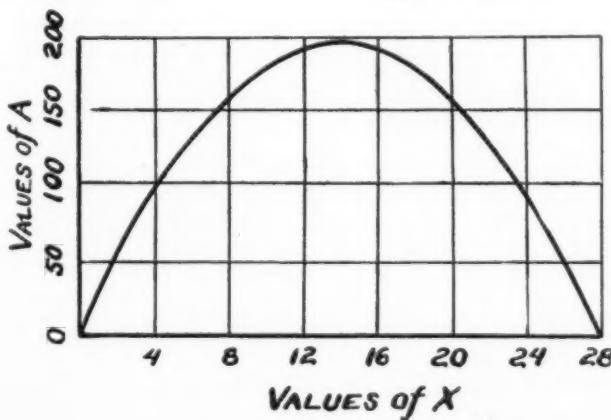
5. *Applications.* (a) *Maximum and Minimum.*—Maxima and minima problems are always stimulating even though they are worked out by cumbersome methods. Suppose we have a wire 56 ft. long with which we wish to fence in a plot of ground. What length and width will give the largest area?

Length + width = $1/2 \times 56 = 28$. Therefore we must find two numbers whose sum is 28 and whose product is the largest possible.

$$\begin{aligned}4 \times 24 &= 96 \text{ sq. ft.} \\8 \times 20 &= 160 \\12 \times 16 &= 192 \\13 \times 15 &= 195 \\14 \times 14 &= 196 \\15 \times 13 &= 195 \\16 \times 12 &= 192\end{aligned}$$

The dimensions which give the largest area are 14×14 . Now let us work this out algebraically and graphically.

$$\begin{aligned}x &= \text{width} \\28 - x &= \text{length} \\x(28 - x) &= A \\28x - x^2 &= A\end{aligned}$$



From the graph we find that the curve continually rises until it reaches its maximum $x = 14$, $A = 196$, and then decreases correspondingly. Let us differentiate the function and see what happens.

$$dA = (28 - 2x)dx.$$

$dA/dx = 28 - 2x$ which on the graph is 0 at the minimum value of A and which means that the slope of the curve is 0 at this value. So, setting $dA/dx = 0$ and solving for x , we have

$$\begin{aligned}28 - 2x &= 0, \\x &= 14, \\28 - x &= 14.\end{aligned}$$

Other examples which are less easily solved by arithmetic should be given here so that the student may realize the superiority of the calculus method. For instance,

(1) Find the maximum rectangle that can be inscribed in a circle.

(2) Find the volume of the largest right circular cone which can be inscribed in a sphere of diameter 15 in. ($V = \frac{1}{3}\pi r^2 h$).

(3) Divide 24 into two parts so that the product of the square of one part multiplied by the other part may be a maximum.

(4) Find a general rule which will apply to any problem like the above. (Ans. Number must be divided in ratio of 2:3.)

Similar problems dealing with minimum values may be given and through graphic representation the student may be shown that maximum and minimum mean greatest and least in a given vicinity and that there may be more than one value for each in the given curve.

(b) *Rate of change.*—In the equation $S = \frac{1}{2}gt^2$, we have found that the velocity or speed is not uniform but is constantly changing.

32 ft./sec. at end of first sec.

64 ft./sec. at end of second sec.

96 ft./sec. at end of third sec., etc.

The rate of change then is 32 ft. per sec. or a uniform change in the speed.

Let us plot the curve $S = \frac{1}{2}gt^2$ as we did before and beside it the curve $dS/dt = 32t$.

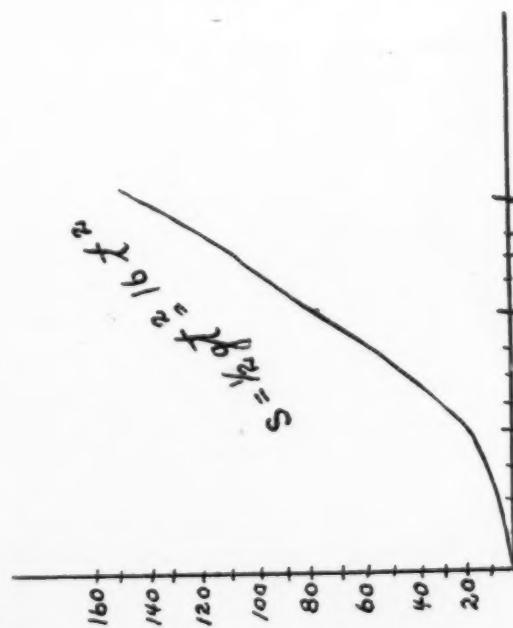
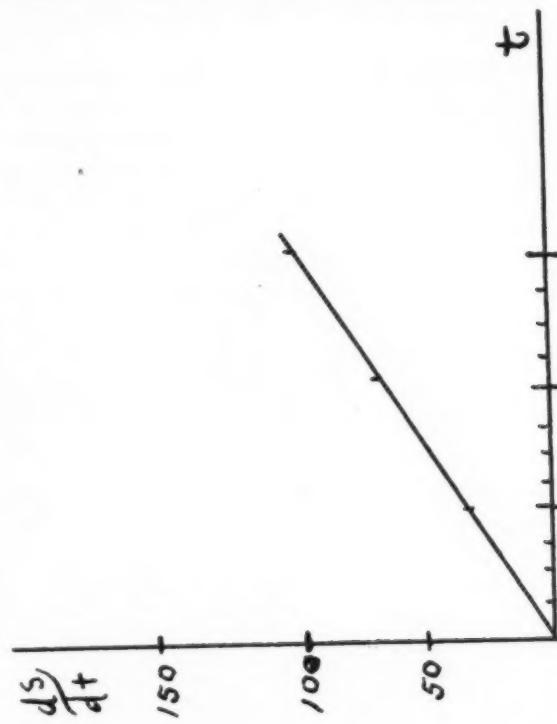
In the first graph we see that the slope of the curve (which we have found to be the same as the speed) is changing at every point of the curve or at every instant of time; in the second curve, the rate curve, we see that the rate of the change in the speed is constant since the slope of the curve is constant.

Examples:

(1) The area of a circular piece of metal is expanding by heat. Compare the rates of increase of area and radius.

(2) A cube of 5 in. edge is expanding. The edge is increasing at the rate of .003 in. per min. At what rate is the volume increasing?

(3) A spherical rubber bag is pumped with air at the rate of 8.4 cu. in. per sec. Find the rate of increase of its diameter when it measures 6 ft.

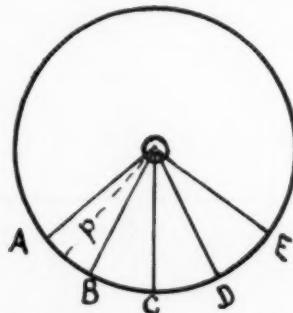


It may be possible that students will find acceleration or rate of change in velocity confusing at first. In such an event it would be advisable to postpone it for a later date—until they take it up again in the infinitesimal calculus.

(b) Integration

1. *Summation*.—A kind of addition.

Integration, as the students might guess from its derivation, may be defined as a putting together of the various parts of a whole to make it complete, *i.e.*, it may be represented as a special kind of addition, the reverse of differentiation which we found to be a special kind of subtraction.



Let the circle O be cut into a large number of equal triangular strips. Draw AB , BC , CD , etc., to form the triangles AOB , BOC , COD , etc.

$$\text{Area of } AOB = \frac{1}{2}AB \cdot a,$$

$$BOC = \frac{1}{2}BC \cdot a,$$

$$COD = \frac{1}{2}CD \cdot a, \text{ etc.}$$

Area of all the triangles is

$$\frac{1}{2}AB \cdot a + \frac{1}{2}BC \cdot a + \frac{1}{2}CD \cdot a + \dots \text{ or}$$

$$\frac{1}{2}a(AB + BC + CD + \dots) = \frac{1}{2}a(\text{perimeter of polygon } ABCD \dots).$$

Now let the sides of the polygon and consequently the number of triangles increase indefinitely. Then the sum of the areas of triangles will approach the area of the circle as a limit and the perimeter of the polygon approaches the circumference of the circle as a limit. Hence,

$$\text{Area of circle} = \frac{1}{2} \text{ circumference} \times \text{radius.}$$

It is quite possible that the students may have forgotten their plane geometry underlying the above discussion, in which case it would be well to review a few propositions which have to do with the theorem of limits.

2. *Summation—Limit of a Sum.*—Using our notations of differentiation, we may express this problem thus:

Let the area of each little triangle be denoted by ΔA and each little bit of perimeter by ΔP , then

$$\Delta A = \frac{1}{2}a \Delta P.$$

Adding together all these little ΔA 's, we have

$$\text{Sum of } \Delta A\text{'s} = \text{sum of } \frac{1}{2}a \Delta P\text{'s.}$$

Now let the number of ΔA 's increase indefinitely and we see that

$$\text{Limit of sum of } \Delta A\text{'s} = \text{area of } \odot.$$

Putting this idea in more convenient form,

$\lim \Delta A = \lim \frac{1}{2}a \cdot \Delta P$ as ΔA and ΔP approach 0,
 $\int \Delta A = \frac{1}{2}p \int \Delta P$, using a kind of "S" to denote the
 limit of the sum of the tiny triangles or elements.

$$\text{Then } A = \frac{1}{2}pC.$$

3. *Derivation of Formula.*—It might be of interest at this stage to go back to our first examples in differentiation and see if we can obtain a general rule for finding the whole when a little bit or element is given.

$$\begin{array}{ll} A = x^2, & A = \pi r^2, \\ dA = 2xdx, & dA = 2\pi r dr, \\ L = 39T^2dt, & y = ax^n, \\ dL = 78Tdt, & dy = anx^{n-1}. \end{array}$$

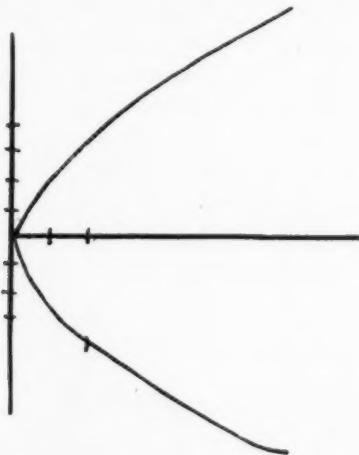
From these formulas we find that in order to get y from dy , A from dA , etc., we increase the exponent by one and divide the coefficient by the given exponent, or

$$y = \frac{anx^{(n-1)+1}}{n} = ax^n.$$

At this point the students should have practice in solving simple problems involving the above general formula.

Examples:

(1) Find the area enclosed by the curve $y^2 = 2x$ and the x -axis. Suppose we let this area be divided up into very thin slices whose height is y and whose width is dx . Each one of



these tiny slices is very nearly a rectangle whose area is ydx . Let us take the limit of the sum of these rectangles as they increase indefinitely and we will have

$$\int dA = \int y dx,$$

but

$$y = \sqrt{2x^{1/2}dx}; \quad \therefore \quad \int dA = \sqrt{2} \int x^{1/2} dx,$$

$$A = \frac{2\sqrt{2}}{3} x^{3/2}.$$

Since $\int dA$ is the sum of the ordinates of the curve, A , the area under the curve must be equal to the sum of the ordinates.

To make this method more general, let us take the general equation $y = f(x)$, then when we integrate as above we will have for

$$\int dA = \frac{2\sqrt{2}}{3} x^{3/2} dx, \quad \int dA = \int f(x) dx, \quad \text{or} \quad A = \int f(x) dx.$$

It would be a very good exercise for the student to derive this formula step by step as has been done in the example above. Then the student should be given or should make up similar examples for drill in these new ideas.

4. *Definite Integral.*—Now if we wish to find only a part of the area under a curve, we may integrate between definite limits. Suppose we are required to find the area in the above curve between the points a and b , any two points chosen on the curve. Obviously the area would be equal to the sum of the ordinates between these two points, a and b , or using the calculus notation,

$$A = \int_a^b f(x)dx = F(x) \Big|_a^b = F(b) - F(a).$$

The student should be shown this graphically and should be given problems involving the integration of plane figures as the circle, ellipse, parabola, etc.

The integration of solid figures may be taken up now in the ordinary calculus way by getting an element of volume and integrating. Illustrations and examples may be found in almost any text and the simpler ones (as volume of a solid with parallel bases and the volume of a solid of revolution such as the student has had in plane geometry) may prove interesting from the calculus point of view. He, of course, should also be given some problems that would be difficult or impossible to do without the calculus in order that he may appreciate the power of the calculus method.

D. The Infinitesimal Calculus

If the student has grasped the fundamental notions of the calculus as presented in the foregoing discussion, he is now ready for a simple course in the infinitesimal calculus. It is planned that about one semester will be spent on the introductory work (providing, of course, that the student has been previously well grounded in the first three years' work as described in one of the plans suggested on pp. 4 and 5); the second semester may be spent on such topics as are taken up in any course in calculus and as time allows. Among such topics may be included the following:

1. Differentiation of Algebraic Functions.
 - a. Polynomial.
 - b. Product.
 - c. Quotient.
 - d. Power.
2. Differentiation of Transcendental Functions.
 - a. Trigonometric Functions.
 - b. Logarithmic.

3. Integration.
 - a. As in Introduction.
 - b. Applications from Geometry and Physics.
 - c. Length of Plane Curve.
 - d. Area of Surface of Revolution.
4. Indeterminate Forms $0/0$ and ∞/∞ .
5. Differential Equations (very simple ones).

III. CONCLUSION

We have seen in this discussion that the calculus is the logical outgrowth of present day tendencies in the reorganization of mathematics courses in secondary education and in the improvement of mathematical instruction; that schools abroad corresponding to our high school have been teaching it successfully for several years; and that those public high schools and experimental schools in this country that have introduced such a course have reported success.

Since well-trained teachers are becoming more numerous and competition is growing keener among them, soon there will be a supply—for the stronger high schools at least—competent to teach the elementary calculus in secondary schools. As for the importance of the calculus, let us reiterate that science, engineering, and industry are demanding more and more calculus and that America, the greatest manufacturing and engineering nation in the world, needs more trained mathematical minds; as a powerful mathematical tool, it shows the student the far-reaching influences of mathematics upon which the development of civilization has always been dependent; it develops a kind of thought in its dealings with small changes in related quantities that is useful in considering everyday problems.

The subject matter and method of presentation as outlined in this treatise need not necessarily be adhered to, for this is but one of many possible courses for secondary schools. However, one general rule must prevail: to make the course successful, the calculus must be presented to the student as a continuation of his previous mathematical training and must be related to his experience in a commonsense way.

UNIFORM GRADING OF EXAMINATIONS IN ALGEBRA

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About fifteen years ago investigators began calling attention to the great variation in grades that different teachers give an examination paper. If, for example, the pupil is solving the equation $5x - 3(x - 2) = 14$ and obtains $5x - 3x - 6 = 14$ and then $x = 10$, shall this work be marked zero because the solution is incorrect or shall the pupil be given some credit for his ability to solve $5x - 3x - 6 = 14$ even though the pupil has made an error in obtaining this equation? The object of this paper is to explain a method by which different teachers can grade an examination paper in algebra and all agree on the score.

In a discussion of this question at a meeting of the Chicago Men's Mathematics Club various interesting points of view were presented. The score on the above work would vary not only with different teachers but even a single teacher might well assign different grades to the above work depending on what the object of the test was and when it was given. Examinations are used for many purposes. If the class has just been studying the use of parentheses, a test may be given not so much for the benefit of the class but for the teacher's benefit; that is, the teacher may wish to know how many pupils have mastered the topic and which pupils need further attention. In such a test the number of errors made in the removal of the parentheses is of chief importance, and the right or wrong solution of the resulting equation is of secondary importance.

On the other hand, this particular equation may have arisen when the teacher was giving a test on verbal problems in which the teacher's main object was to see if the pupil could derive correctly the equation on which the problem depends. Here the derivation of the equation is of chief importance and the incorrect treatment of the parenthesis may even be regarded as a mere accident, the pupil having previously shown (in other tests) that he knows the right way to use a parenthesis.

Again, the equation may have arisen on a test which aimed to review all that the pupil is supposed to have learned during a period of ten weeks or more. Here *any* error is an indication that the pupil has not acquired the mastery of all the details that he should have. However, an error in the use of parentheses might well be regarded as a serious error during the first semester's work but treated as a slip of the pen if it happened on a final examination at the end of the year's work. Even if the pupil checks the solution, it is possible to make an error in the check, thus raising also the question of the importance and value of the check in determining a grade.

To overcome these objections teachers have tried tests each of which covers only one process. Thus the teacher gives a test in which the pupil is asked to do nothing more than solve equations like $5x - 3x + 6 = 20$, tests in which the pupil is asked to remove the parentheses from $5x - 3(x - 2) = 14$ and carefully instructed *not* to solve the resulting equation, or tests in which the pupil is asked to change the equation

$$\frac{x}{2} + \frac{x - 9}{5} = \frac{x}{4}$$

into an equation without fractions but *not* to solve the resulting equation. We may have tests in which the pupil when learning a certain process is told "Do the first step in solving the equation $x^2 - x - 6 = 0$ and not more." Obviously the work can then be marked right or wrong, and the pupil's score would not depend on who did the marking.

Tests which thus emphasize a single operation are useful chiefly to the teacher as they show which specific operations need further drill. Such tests do not stimulate

1. The development of the patience and willingness to go through a mass of details, keeping each item in mind.
2. The ability to do close and detailed thinking.
3. The ability to give sustained attention (concentration).

These are three of eleven items mentioned on page 369 of the National Committee Report as "the abilities which the successful study of mathematics encourages and develops." Examinations that would measure these accomplishments would be preferred provided the papers could be graded in an accurate manner. A possible method of doing this for schools in which an exami-

nation is given to many classes and each teacher marks her own papers will be described.

The following examination was given two weeks before the close of the term. The problems do not cover every topic taught during the year; thus, there is no work on radicals or formulas or verbal problems. The test aims at examining the pupils' ability to select and perform correctly a succession of operations in addition, subtraction, multiplication, division, use of parentheses, and equations. Experience showed that 25 minutes was ample time for the test. The work was mimeographed so that no copying from the blackboard was necessary; the directions were so complete that no pupil need ask for any further directions.

The five exercises were:

1. Subtract the two fractions; that is, write as one fraction:

$$\frac{a^2}{10b^2} - \frac{c}{6b}.$$

2. Add the two fractions; that is, write as one fraction (you need *not* check the work):

$$\frac{x-4}{x^2+3x} + \frac{9}{x^2-2x-15}.$$

3. Solve the following set of equations by the *substitution* method (not by any other method). Also, *check* the answers.

$$\begin{aligned} 4x + 7y &= 23, \\ 3x - 5y &= 7. \end{aligned}$$

4. Solve the following quadratic equation by the method of completing the square. No other method will count. You may check the work for your own use but the check does not count on your mark.

$$y^2 - 5y - 6 = 0.$$

5. Solve the following equation, first clearing of fractions, and then solving the quadratic by factoring. No other method will count. You may check the work for your own use but the check does not count on your mark.

$$\frac{x}{x+5} - \frac{x-2}{x^2+5x} = \frac{16}{3x+15}.$$

The directions for marking the papers were as follows:

Assume that each problem consists of a number of distinct and *independent* steps, each of which is either right or wrong. The third problem, for example, may be divided into the following separate problems:

1. Solve $4x + 7y = 23$ for y .

2. Substitute $y = \frac{23 - 4x}{7}$ in $3x - 5y = 7$.

3. Clear of fractions the equation:

$$3x - 5 \frac{(23 - 4x)}{7} = 7.$$

4. Remove the parentheses: $21x - 5(23 - 4x) = 49$, etc.

If now, in working step 1, the pupil obtains $y = \frac{23 + 4x}{7}$,

then for this pupil the second step is really:

$$\text{Substitute } y = \frac{23 + 4x}{7} \text{ in } 3x - 5y = 7,$$

and if the pupil does this work correctly, then for him, step 2 is correct.

If each of the five problems is analyzed in this fashion, there are 50 steps in all; hence for each incorrect step two points are deducted from the score. The number of steps in the problems are 6, 9, 10, 10, and 15 respectively.

Since the number of errors that a pupil can make is almost inconceivable, further analysis of each problem is necessary.

In the first problem, the 6 steps are:

- | | |
|--|---|
| <ol style="list-style-type: none"> 1. Select the C.D.: $30b^2$. 2. $30b^2 \div 10b^2 = 3$. 3. $3 \times a^2 = 3a^2$. | <ol style="list-style-type: none"> 4. $30b^2 \div 6b = 5b$. 5. $5b \times c = 5bc$. 6. Write: $\frac{3a^2 - 5bc}{30b^2}$. |
|--|---|

The pupil is not expected to present the work in the above style, but each error can be traced to one of these six steps. If $60b^3$ or $60b^2$ or $30b^3$ is used as the common denominator, then steps 1 and 6 are incorrect but not the other steps. And step 6 may be correct if the final fraction is reduced to the form shown

above. If the answer is written $3a^2 - 5bc$ (the denominator missing), then step 6 is incorrect but not the other steps. If the answer is $6a^2 - 10bc$ or $3a^2b - 5b^2c$ or $6a^2b - 10b^2c$, then step 1 is wrong and step 6 is wrong but the multiplications and divisions in the other steps are correct. If the pupil obtains the correct answer in step 6 and then does some further incorrect cancellation, then count step 6 as wrong. If the pupil writes a + sign in the numerator of his answer, then step 6 is wrong. Thus all the various errors may be analyzed.

In the second problem there are 9 steps:

1. Factor $x^2 + 3x = x(x + 3)$.
2. Factor $x^2 - 2x - 15 = (x - 5)(x + 3)$.
3. Select the C.D.: $x(x + 3)(x - 5)$.
4. $x(x + 3)(x - 5) \div x(x + 3) = x - 5$.
5. $(x - 5)(x - 4) = x^2 - 9x + 20$.
6. $x(x + 3)(x - 5) \div (x - 5)(x + 3) = x$.
7. $9 \times x = 9x$.
8. Write: $\frac{x^2 - 9x + 20 + 9x}{x(x + 3)(x - 5)}$.
9. Simplify to: $\frac{x^2 + 20}{x(x + 3)(x - 5)}$.

If $(x^2 + 3x)(x^2 - 2x - 15)$ is used as the C.D., count steps 1, 2, and 3 wrong, but steps 4 to 8 are correct (if correctly done). Since the pupil will be unable to obtain the answer in the form shown in step 9, count step 9 as wrong. If the denominator is omitted from either or both steps 8 and 9, count one of the steps wrong, not both. Count one step wrong if the pupil stops with step 8. Errors in signs and multiplications and in factoring can be traced to one of the steps. If, for example, the pupil factors $x^2 - 2x - 15$ as $(x + 5)(x + 3)$, then step 2 is incorrect but steps 3 to 8 may be correct for this pupil; this pupil would be penalized for steps 2 and 9. To get the correct attitude in marking, assume that the questions were:

Factor $x^2 + 3x$.

Factor $x^2 - 2x - 15$.

Find the common denominator of the factors found in the previous steps.

Divide your common denominator by the factors of the first quantity, etc.

In the third problem the 10 steps are:

- | | |
|--------------------------------------|-----------------------------------|
| 1. $y = \frac{23 - 4x}{7}$, | or $x = \frac{23 - 7y}{4}$, |
| 2. $3x - 5\frac{(23 - 4x)}{7} = 7$, | $3\frac{(23 - 7y)}{4} - 5y = 7$, |
| 3. $21x - 115 + 20x = 49$, | $69 - 21y - 20y = 28$, |
| 4. $41x = 164$, | $- 41y = - 41$, |
| 5. $x = 4$, | $y = 1$, |
| 6. $16 + 7y = 23$, | etc. |
| or $12 - 5y = 7$, | |
| or $y = \frac{23 - 16}{7}$. | |
| 7. $7y = 7$, | |
| or $- 5y = - 5$. | |
| 8. $y = 1$. | |
| 9. Check: $16 + 7 = 23$. | |
| 10. Check: $12 - 5 = 7$. | |

If the value of y is obtained by the third of the methods in step 6 and solution is checked as in step 10, the check should be considered correct even though step 9 is not shown (because this method of checking is taught in some classes). A pupil may do one or more of the steps mentally without penalty. If a pupil obtains

$$y = \frac{23 + 4x}{7}$$

in step 1, then steps 2 to 8 may be correct for this pupil, but steps 1, 9, and 10 are incorrect.

In the fourth problem the 10 steps are:

1. $y^2 - 5y = 6$.
2. Think: $\frac{1}{2}$ of 5 is $\frac{5}{2}$; $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$.
3. $y^2 - 5y + \frac{25}{4} = 6 + \frac{25}{4}$.
4. $6 + \frac{25}{4} = \frac{49}{4}$.
5. The square root of the left member is $y - \frac{5}{2}$.
6. The square root of the right member is $\pm \frac{7}{2}$.

7. $y = \frac{5}{2} + \frac{7}{2}$.

8. $y = 6$.

9. $y = \frac{5}{2} - \frac{7}{2}$.

10. $y = -1$.

If step 2 is wrong and the pupil adds, for example, $\frac{25}{2}$, but the subsequent work is correct, the final answer being

$$y = \frac{5}{2} \pm \sqrt{18\frac{1}{2}},$$

then count steps 2, 8, and 10 as wrong. If the pupil writes $y + \frac{5}{2} = \pm \frac{7}{2}$, then step 5 is wrong. If any two errors are of such a nature that they correct each other, then the two steps are incorrect but the final answer is correct. If the pupil thinks that $\frac{5}{2} + \frac{7}{2} = \frac{12}{4}$, then step 8 is incorrect but not step 7.

In the fifth problem the 15 steps are:

1. Factor: $x^2 + 5x = x(x + 5)$.
2. Factor: $3x + 15 = 3(x + 5)$.
3. Select the multiplier: $3x(x + 5)$.
4. $3x(x + 5) \div (x + 5) = 3x$.
5. $3x \times x = 3x^2$.
6. $3x(x + 5) \div x(x + 5) = 3$.
7. $3 \times (x - 2) = 3(x - 2)$.
8. $3x(x + 5) \div 3(x + 5) = x$.
9. $16 \times x = 16x$.
10. Write: $3x^2 - 3(x - 2) = 16x$,
11. $3x^2 - 3x + 6 = 16x$,
12. $3x^2 - 19x + 6 = 0$.
13. Factor: $(3x - 1)(x - 6) = 0$.
14. $3x - 1 = 0, \quad x = \frac{1}{3}$.
15. $x - 6 = 0, \quad x = 6$.

Again the pupil need not show every step. Thus steps 5, 7, and 9 ask: Do you know how to find the new numerators when adding fractions, or the correct use of the multiplier? If step 11 is $3x^2 - 3x - 6 = 16x$, then steps 11, 13, 14, and 15 are incorrect.

(Note here that we have one possible answer to the question raised in the first paragraph: What penalty shall be applied to the incorrect use of a parenthesis.) If the wrong multiplier is used, such as $x(x + 5)(3x + 15)$, the equation in step 10 will be a cubic, and it is not likely that the pupil can solve it. Hence count steps 1, 2, 3, 13, 14, and 15 as wrong. Do not penalize the pupil for neglecting to write " $= 0$ " in steps 12 and 13. Many teachers will disapprove of the fifth problem thinking it is more complex than the ninth grade requires. However, in solving fractional equations all teachers agree that the denominators should first be factored and a suitable common denominator then chosen. Hence it seems ridiculous to teach such a rule and then present the pupil with equations in which the denominators cannot be factored. Either the rule should be omitted or else the denominators should be such that the pupil must factor them.

When a test of this nature is given to many classes in the same school, it is essential that the problems be varied so that the pupils in an afternoon class cannot get advance information from a class that meets in the morning. At the same time the problems for the various classes must all be of uniform difficulty. In the first problem, for example, the denominator 30 is not the product of the common denominators but is half the product. Hence in any equivalent problem the common denominator should also be half the product of the denominators. Likewise in variations of the second problem the factoring should all be of the same degree of difficulty with one factor involving a plus sign and one a minus sign. And in the third problem the numerical work should not involve numbers requiring far different arithmetic calculations. The following variations of the five problems are suitable:

In the first problem:

$$\frac{r^2}{10s^2} - \frac{t}{4s}; \quad \frac{r^2}{10t^2} - \frac{v}{8t}; \quad \frac{a^2}{12x^2} - \frac{b}{10x}; \quad \frac{m^2}{8n^2} - \frac{p}{6n}.$$

In the second problem:

$$\begin{array}{ll} \frac{x-3}{x^2+5x} + \frac{10}{x^2-2x-35}; & \frac{x-3}{x^2+2x} + \frac{8}{x^2-3x-10}; \\ \frac{x-2}{x^2+3x} + \frac{9}{x^2-4x-21}; & \frac{x-5}{x^2+2x} + \frac{12}{x^2-5x-14}. \end{array}$$

In the third problem:

$$\begin{cases} 3x + 7y = 13, \\ 5x - 4y = 6, \\ 7x + 2y = 16, \\ 5x - 3y = 7, \end{cases} \quad \begin{cases} 5x + 2y = 19, \\ 4x - 3y = 6, \\ 2x + 5y = 18, \\ 3x - 4y = 4. \end{cases}$$

In the fourth problem:

$$\begin{array}{ll} x^2 - 3x - 10 = 0, & x^2 - 7x - 8 = 0, \\ x^2 - 5x - 14 = 0, & x^2 - 3x - 18 = 0. \end{array}$$

In the fifth problem:

$$\begin{aligned} \frac{x}{x+2} - \frac{x-5}{x^2+2x} &= \frac{11}{3x+6}, \\ \frac{x}{x+4} - \frac{x-3}{x^2+4x} &= \frac{5}{2x+8}, \\ \frac{x}{x+3} - \frac{x-2}{x^2+3x} &= \frac{7}{2x+6}, \\ \frac{x}{x+2} - \frac{x-3}{x^2+2x} &= \frac{15}{4x+8}. \end{aligned}$$

Opinions may differ as to how great should be the penalty for an error that leads to an incorrect solution of an equation. But in any event if a problem is analyzed as shown here and if some agreement is made about the penalty for each kind of error, it is easy enough for different teachers to grade papers uniformly.

A STUDY OF THE ERRORS MADE IN A NINTH YEAR ALGEBRA CLASS

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For a period of three months, I recorded as far as possible the errors made by a beginning class in algebra, in the East Side High School of Madison, Wisconsin. These errors occurred in oral and written class work, and in tests; it was difficult to record errors made in recitation work, without interfering with the progress of the class; further than this, all errors would not necessarily come to my attention.

The result of this study would seem to justify the statement that many of the difficulties ordinarily attributed to algebra are in reality difficulties of a more deep-rooted nature. It would apparently seem as if certain fundamental processes, particularly those of arithmetic, have not become automatic enough for the poorer students to devote their entire attention to whatever is new to them in algebra. The errors noted, which occurred most frequently, were those due to a lack of a thorough enough knowledge of the fundamentals of arithmetic, and those due to faulty reading.

The class under observation was rated as a "B" section in the department. The members of the class, with the exception of three who entered late, had taken the Otis intelligence test; the results of this test gave the highest intelligence score as 135, the lowest 102, the median score 110. On the whole, these students were serious about their work, and interested in doing well. On the basis of their success in class, they might perhaps be classified into five divisions: Group I, the most successful division; Group II, the next most successful division, and so on.

The greatest number of errors noted were caused by incorrect arithmetical combinations. With the exception of VF, Group I, practically every member of the class made mistakes in computation. But the pupils in the lowest group made this type of error more frequently than did the pupils in other groups.

My impression is, that combinations where 9 and 7 occurred caused most difficulty, and that of the fundamental processes, subtraction and to a lesser degree division, caused most difficulty. This latter difficulty was carried over into the work in algebra; subtraction of polynomials gave the class more difficulty than addition, multiplication, or even division.

The following are typical arithmetical errors made by the class: $63 + 13 = 74$; $48 + 56 = 94$; if $x = 11$, $x + 3 = 18$; $48 - 8 + 9 = 48$; $15 - 8 = 6$; $27 - 15 = 11$; $11 - 5 = 16$; $400 - 9 = 331$; $46 - 4 = 40$; $9 \times 7 = 56$; $4^2 = 8$; $\sqrt{16} = 8$; 135×100 could not be obtained mentally; $24 \div 8 = 8$; $8 \div 4 = 8$; $12 \div 4 = 8$; $1 \div 1 = 0$; $39 \div 13 = 2$; and $17 \div 1 = 1$.

Apparently these difficulties were due to lack of sufficient drill. Usually as soon as a student's attention was called to an error of this kind, it was immediately corrected. However, there was one exception to this statement. AW, Group V, said "13 - 9 = 5." When corrected, she said, "Then it's 6."

Fractions gave difficulty; decimals to a lesser extent. The text used contained fewer decimal examples, which may possibly be an explanation. There was always difficulty in checking an equation, when the answer came out a fraction. Typical fractional errors follow: $5 \div 4 = 1\frac{1}{5}$; $5 \div 4 = 4\frac{1}{5}$; $6 \div (-18) = -3$; $1\frac{1}{2} \cdot 1\frac{1}{2} = 1$; $2\frac{2}{3} \cdot 2\frac{2}{3} = 1\frac{1}{3}$; $25S = 10$, $S = 5/2$; and $X/2 \cdot X/2 = X^2$.

RH was especially weak in fractions. The first part of the year, he worked in Group V, and then gradually moved up to Group III. He had taken no intelligence test, but if he had, he would probably have rated low; he was sixteen years old, and a freshman in high school. Yet, he was capable of fair work, because he possessed enough determination. He asked after class one day to have explained to him, why $6/11 \div 6/11 = 16$. Another day after class he asked to have explained to him, how to multiply fractions. In test I, his paper showed the following work on the question to evaluate $12(b/2 - c/3 + d/4)$, if $b = 8$, $c = 2$, and $d = 3$.

$$\begin{array}{rcl} 12(8/2 - 2/3 + 3/4) = & 32/2 & 96/6 \\ 32/2 - 38/3 + 51/4 = 8 7/12 & - 38/3 & 76/6 \\ & \hline & \\ & 20/3 & 3 1/3 & 8 7/12. \end{array}$$

The next type of error which occurred most frequently was incorrect reading, and as a result of this, incorrect copying. There seemed to be prevalent in the class an idea that all work must be done hurriedly, that accuracy was not important, and as a consequence, errors of this kind resulted. These are typical errors: — read as +; 13 read as 3; z read as y ; 64 read as 60; triangle read as rectangle; $2x - 3$ read as $2x - 7$; and CD^2 read as CD^2 .

In copying, these errors occurred: 7.5 copied as 7.7; 17 copied as 11; $4C$ copied as CC ; $3y$ copied as $8y$; and 30 copied as $3A$.

The pupils who consistently made errors of this kind were LG, Group II, and DG of Group IV; both were nervous in temperament, which may partially account for this difficulty. DB, Group IV; EI, Group III; BG, Group II; AW and SW, Group V, also made this type of error frequently. With the exception of LG and BG, the children who made this type of error most often were all rural school entrants.

In oral work, this was an exceptionally easy error to correct, and its consequences were not seemingly of much importance. In problem work, an error of this type caused much more difficulty. LG in solving a written problem set the statement equal to 36, instead of 30. The figure 36 was printed in the next problem. She wasted considerable time, because in going over her work she could find no other way to form the symbols, or the equation, and the answers she had, checked for 36. Each succeeding time, in going over her work, she read 36 for 30, and found her mistake only when she asked about the problem in class.

Another type of error, closely allied to incorrect reading of the text, was an error due to the failure on the part of the student to write legibly enough to read his own writing. This particular difficulty did not occur so generally as errors in arithmetical computations, nor so persistently with any one student as the reading difficulty; but, when it did occur, it caused enough trouble. To a beginner, who can solve equations in only one variable, it was confusing to have P appear with D ; m with n ; or R with P in the course of a solution. CY, IB 104, the poorest student in the class, was the chief offender in this respect.

ES, Group I, IB 122, ordinarily a careful student, made this error in a division example of test III. $+ 10a^3$ was to be sub-

tracted; the result was $-10a^3$. She wrote it thus $-10a^3$. Then she considered the result as $+10a^3$, and obtained an incorrect answer.

HG, also a careful student of Group II, a girl with an IB of 115, on the same example, of this same test, test III, wrote $53a$ so that it looked like $50a$; she considered the value as $50a$, and thus obtained an incorrect result.

Difficulty arose, especially in problem work, largely due to a lack of the knowledge of the words used. It would seem to be a general understanding of the terms employed that brought about the success of the better students in the class. AF, IB 126; AO, IB 135; VF, all of Group I, grasped the meaning of the words used very readily. LG would not have belonged to Group II, with her reading handicap, if it were not for the fact that she, too, knew the meaning of words. The following are typical difficulties which arose, largely due to the lack of knowledge of words.

I. There was difficulty in distinguishing between the meaning of expressions like 3 more, and three times more.

II. In examples in subtraction, there was little difficulty in obtaining the correct result, if the minuend was stated first. In an example such as this, "Subtract $3x - 4y + z$ from $5x - 4y - 8z$," the students were not always careful to consider that the first polynomial was the subtrahend, but treated it as the minuend.

III. Diminished was a new word to some of the poorer students. $(3 - y)$ was to be diminished by $(9 - y)$. Six members of the class added the quantities. They were accustomed to the more usual expression, "subtracted from."

IV. Excess was another word that troubled some of the pupils in the lower groups. They understood the phrase, one thing exceeds another, but the noun form was unfamiliar.

V. In rectangle problems, a persistent error was made in adding only two sides for the perimeter. This may have been due to the fact that in getting the area of a rectangle, the length and width are used but once.

VI. The word interest, which would apparently be one of the most familiar words, was unknown to GT, Group III. She had had no experience with the term outside of school, and had difficulty in grasping its meaning.

VII. The class recognized consecutive numbers in arithmetic, but in carrying the idea over into algebra they experienced difficulty. They attempted to use a and b as symbols for consecutive numbers, and $1x$ and $2x$.

VIII. Age problems were not popular. Many of the pupils had difficulty in expressing correctly the symbols for terms like "the age of A seven years ago," and "the age of B six years hence." The statement of these problems in the text used is perhaps at times confusing; further than this, the solution of a problem of this type fell outside of the interest and practical experience of these young people. In contrast to this type of problem, time, distance, and rate problems were liked, and little difficulty was found in their solution.

Of the work new to the class, signs gave most difficulty; they gave especial difficulty in connection with parentheses. This was a common error: if $6x - 4(x - 5) = 6$, then $6x - 4x - 20 = 6$. The students in Groups IV and V confused the rules for removing + and - parentheses. They also wished to change the signs of terms outside the parentheses. For instance, $7x + 3 - (5x - 4) = 7x - 3 - 5x + 4$. In connection with parentheses, this type of error occurred, but not to any great extent, $3(a - 10) = 3a - 10$.

There was great variation in the rapidity with which the members of the class could work. In one recitation period, AF and VF, Group I, solved and checked twenty simple equations, while AW and SW, Group V, solved three. AW and SW are sisters. When SW entered the class, she could not distinguish between a plus sign and a multiplication sign.

Exponents caused some slight difficulty. $t \cdot t = t$, and $w^2 \cdot w^2 = w^2$ are typical errors that were made. AF, Group III, contributed $y^8 \div y^4 = y^{2/8}$.

During these three months, the class observed wrote three full period tests, and the papers were analyzed in order to note the types of errors made. The first test was designed to give practice in removing parentheses. Tests II and III were departmental tests. On test II, the class under observation obtained a median score of 77; the other two classes in the school, which might be compared with section "B," obtained median scores of 78 and 80. Similarly, on test III, the median scores were practically the same. Test IV, the final examination, was written at the close of the first semester's work.

Types of Errors	Number of Errors			
	Test I	Test II	Test III	Test IV
Sign errors	27	14	5	18
Failure to check equation or to substitute correctly	3	2	1	0
Errors in copying	16	24	15	19
Arithmetical errors	7	22	20	15
Incorrect operation	0	9	20	11
Incorrect order caused difficulty	0	10	9	17
Sp.-Errors in exponents	0	6	11	3
Errors in symbols in problem work	4	2	13	10
Symbols correct, equation incorrectly formed	9	4	10	9
Equation for problem correctly formed, but not solved correctly	1	0	0	0
Errors Sp.-in removing parentheses	0	0	0	6
Totals	67	93	104	108

The membership of the class under observation remained practically the same for the second semester's study of algebra. RH of Group III, and CE of Group V withdrew from school; VF and AO of Group I, and ET of Group V were transferred to other sections. During the last three months of the year's course, I again made a count of the errors which this class made. As a part of this record, one hundred and forty-one home-work papers handed in by the different members of the class, were studied, and the errors classified. The result of this particular phase of the study follows.

Types of Errors	Number of Errors	Percent of Errors
Errors in simple arithmetic	62	32.0%
Exs. $5 \cdot 22N = 105N$. $5 \cdot 4/5 = 38/5$.		
Errors in signs in transposition	24	12.4%
Ex. $2R + 7 = - 15$, $2R = - 8$.		
Errors in signs in removing parentheses	19	9.8%
Ex. $80x - 15(18 - 8x) = 50$, $80x - 270 - 120x = 50$.		
Errors in addition or subtraction	9	4.6%
Ex. $91 - 35y - 8y = - 38$, $43y = - 129$.		
Errors in signs in division	4	2.1%
Ex. $- 11A = - 22$, $A = - 2$.		

Types of Errors	Number of Errors	Percent of Errors
Errors in signs in clearing of fractions	4	2.1%
Ex. $16x - \frac{54 - 24x}{5} = 10,$ $80x - 54 - 24x = 50.$		
Incorrect operation used	20	10.3%
Ex. $15y = 30,$ $y = 1/2.$		
Errors in extracting square root	14	7.2%
Exs. $\sqrt{8836} = 93.$ $\sqrt{5/3} = 15/3 = 5.$		
Errors in copying	13	6.7%
Ex. $5 - a + 6b = 1$, copied as $5 - a + 66 = 1.$		
Errors in factoring	9	4.6%
Ex. $(C - 2)^2 = (C + 2)(C - 2).$		
Failure to form the correct equation for a written problem	5	2.6%
Errors in completing the square in the solution of quadratics	4	2.1%
Ex. $z^2 - 3z = 8,$ $z^2 - 3z + 9 = 17.$		
Incorrect use of exponents or coefficients	3	1.5%
Ex. $(2x - 5)^2 = 4x - 10.$		
The unknown in the answer	3	1.5%
Ex. $S^2/4 - S^2 = A,$ $S^2 - 4S^2 = 4A,$ $S = \pm \sqrt{4S^2 - 4A}.$		
Error in removing parentheses	1	.5%
Totals	194	100.0%

In addition to this, I noted the errors which occurred in class work. My impression is, that difficulty arose again because of a failure on the part of the student to comprehend the meaning of words. For instance, GT failed in response to the question, "How many common points do the graphs have?" because she did not know the meaning of the word common. AF, HA, and MS, students in the lower groups, had difficulty with signs; they had most difficulty with signs in transposing terms, and in connection with fractional expressions. The occurrence of fractions was always a sign for trouble; fractions were especially troublesome in graph work, and the solution of quadratics. One hundred and seven definite errors were noted, which were classified as follows.

Types of Errors	Number of Errors	Percent of Errors
Arithmetical errors	45	42.1%
Exs. $4^2 = 8$. $1/4 \cdot 1/4 = 2 1/4$. $2 \cdot 0 = 2$. $48 + 6 = 53$.		
Errors in copying or reading	28	26.2%
Ex. One fraction from two different examples was copied as,		
$\frac{m^2 - 7mn + 12n^2}{m^2 - 9mn + 20n^2} \cdot \frac{5x^2 - 45y^2}{4x^2 + 4xy - 24y^2}$.		
Errors in signs	9	8.4%
Ex. $2a - b - 3 = -6$, $2a - b = -9$.		
Failure to perform the same operation on both sides of the equation	6	5.6%
Ex. $\begin{array}{r} 27x - 24y = 18 \\ 64x - 24y = 56 \\ \hline -37x = 74 \end{array}$		
Failure due to a lack of the knowledge of words	6	5.6%
Failure to complete solution	5	4.7%
Ex. $-6p = 10$.		
Incorrect operation	3	2.8%
Ex. $1/2c = 2$, $c = 1$.		
Errors in clearing of fractions	3	2.8%
Ex. $\frac{3(28a - 7)}{2} = 42a - 21$.		
Incorrect use of exponents or coefficients	2	1.8%
Totals	107	100.0%

The class wrote the following short test on square root and surds:

- I. Simplify, $\sqrt{48} - 2\sqrt{3} + \sqrt{300}$.
- II. Simplify, $\sqrt{2/9} + \sqrt{1/8} - \sqrt{1/2}$.
- III. Extract the square root of 43264.
- IV. Extract the square root of $x^4 - 6x^3 + 13x^2 - 12x + 4$.

The errors made on this test fall into the following classification:

Types of Errors	Number of Errors	Percent of Errors
Failure to add fractional coefficients correctly	6	40.0%
Ex. $\sqrt{2}/9 + \sqrt{1}/8 - \sqrt{1}/2 =$ $1/3\sqrt{2} + 1/4\sqrt{2} - 1/2\sqrt{2} = 5\sqrt{2}$.		
Simple arithmetic	4	26.6%
Ex. $16 \times 1.7332 = 16.784$.		
Incorrect operation	2	13.3%
Work left in incomplete form	1	6.7%
Removed radical sign, without extracting root	1	6.7%
Sign error	1	6.7%
Totals	15	100.0%

The class liked the work in graphing simple statistics, and simple simultaneous equations. As LG expressed it, "This is the most fun yet. Why you don't need any brains to do this." The class wrote the following test on graphing and simultaneous equations:

- I. Graph $2x - y = 4$,
 $2x + 3y = 12$.
- II. Graph $2x + y = 4$,
 $6x + 3y = 12$.
- III. Graph $2x + 3y = 6$,
 $4x + 6y = 8$.
- IV. Temperature readings were given for nine consecutive hours, from which a temperature chart was to be drawn.
- V. Solve by the addition and subtraction method:

$$\begin{aligned} 15a + 8b &= 1, \\ 10a - 7b &= -24. \end{aligned}$$

The errors made are classified as follows:

Types of Errors	Number of Errors	Percent of Errors
Errors in substitution of values	8	42.1%
Ex. $4x + 6y = 8$, If $x = 0, y = 8$.		
Failure to state legend	4	21.1%
Errors in correctly plotting points	2	10.5%
Sign errors	2	10.5%

Failure to perform the same operation on both sides

of the equation involved 2 10.5%

$$\text{Ex. } 30a + 16b = 2$$

$$30a - 21b = -72$$

$$\underline{- \quad 5b = -70}$$

Error in copying 1 5.3%

Totals 19 100.0%

At the end of eight months, the class wrote the Hotz First Year Algebra Problem Scale, Series A. Their median of achievement was 5.3; the lower quartile for the class was 4.2; the upper quartile was 6.5. The tentative median of achievement for a nine months group is 5.6. On this test, the class made the following errors.

Types of Errors	Number of Errors	Percent of Errors
Failure to comprehend the conditions of the problem	27	62.8%
Incorrect operation as $m + r$, instead of $m - r$	9	20.9%
Work incomplete	4	9.3%
Incorrect reading as $890/v$, instead of $980/v$	2	4.7%
Conditions of problem understood. Failure to use parentheses correctly. For problem 9, this result was given, $b - 12.a - 4 = b.a$	1	2.3%
Totals	43	100.0%

At the end of nine months the class wrote the Hotz Equation and Formula Scale, Series A. Their median score was 8.2; the lower quartile for the class was 7.2; the upper quartile 8.9. The tentative median of achievement for a nine months group for this particular test is 7.8.

The following table shows the errors the class made on the Hotz Formula and Equation Scale.

Types of Errors	Number of Errors	Percent of Errors
Arithmetical errors	12	41.4%
Exs. $1/2 \cdot 10.8 = 1/2 \cdot 18 = 9$ $5m = 100,$ $m = 4.$		
Errors in clearing of fractions caused by errors in special products	5	17.2%

Types of Errors	Number of Errors	Percent of Errors
Errors in use of fundamental law of algebra	5	17.2%
Ex. $c - 2(3 - 4c) = 12$ considered as $(c - 2)(3 - 4c)$.		
Errors in signs in division	2	6.9%
Ex. $-4x = 2$, $x = 1/2$.		
Incorrect operation	2	6.9%
Ex. $2x = 4$, $x = 1/2$.		
Errors in signs in transposition	2	6.9%
Ex. $5a + 5 = 61 - 3a$, $2a = 56$.		
Error in solution of quadratic equation	1	3.5%
Ex. $p^2 - 5p = 50$, $p(p - 5) = 50$, $p = 0$, $p = 5$.		
Totals.....	29	100.0%

A total number of 407 definite errors were recorded, during the period of the second three months that the class was under observation. Of these mistakes, the largest number 144, 35.4% of the total count, were errors in simple arithmetic; the next largest number of mistakes, 76, or 18.7% of the total count, were errors in signs; there were 44 errors in copying or reading, 10.8% of the total; the use of the incorrect operation caused 42 errors, or 10.3%; the lack of understanding of terms brought about 38 failures, or 9.3% of these mistakes. Summarizing, of the 407 errors, 344 or 85.4% of the mistakes, were errors in simple arithmetic, signs, copying or reading, in using the incorrect operation, and errors due to a lack of comprehension of the meaning of words.

MATHEMATICS IN THE JUNIOR HIGH SCHOOL*

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No change in educational administration or in educational policy in the last fifty years has offered so great an opportunity for improvement in high-school mathematics as the organization of the junior high school. But as we felicitate ourselves upon the possibilities we need to remember that many of the changes now taking place were advocated by leaders thirty years ago. There is a striking similarity between some recommendations of the National Committee on the Reorganization of Mathematics in Secondary Education published in 1923 and the Report of the Commission on Mathematics of the Committee of Fifteen in 1893. There were seers on that Commission headed by Simon Newcomb and we are just catching up with their vision. That educational administration has made this advance is due to the study and appreciation of the varying needs and abilities of pupils and the demands of a scientific age. That mathematics teachers in 1926 are accepting changes advocated thirty years ago is due in large part to the work of the International Commission on the Teaching of Mathematics and the work of the National Committee. Every teacher of secondary mathematics should own and read the Report of the National Committee, the best text that I know on the teaching of the subject; and every one should know of the significance of the work of the International Commission.

The aims and spirit of junior high school mathematics has received its best expression in the Report of the National Committee:

"In the junior high school, comprising grades seven, eight, and nine, the course for those years should be planned as a unit with the purpose of giving each pupil the most valuable mathematical training he is capable of receiving in those years, with little reference to the courses which he may or may not take in succeeding years."

* Read at the Conference of Teachers of Mathematics at the University of Iowa, October 9, 1926.

Furthermore, as the Report goes on to say, this plan in no way conflicts with the interests of the students who continue the study of mathematics; and it is in accord with the purposes of the junior high school one of whose chief aims is to offer a programme of studies that will aid the pupil in orienting himself, aid him in discovering his likes and capabilities.

What then is the most valuable mathematical training that pupils are capable of receiving in grades seven, eight, and nine? What would you want the junior high school to do for you if you were entering the seventh grade? Here are some things it should give.

Better understanding of the meanings of the fundamental processes of arithmetic.

Ability to compute with a fair degree of accuracy and speed.

Useful information of the mathematics of the home, community, shop, and store.

A considerable body of knowledge from the domains of algebra and geometry, to be used in interpreting and extending experience, and a mastery of some of the elementary methods of those subjects.

Methods of attacking problems.

Appreciation of the conciseness, accuracy, certainty, and elegance of mathematical symbolism and argument.

A vision of the possibilities of mathematics and of its place in human culture.

Of course it is not possible in three years to take our classes far in all of these directions. But they may be taken a considerable distance in some and given a start in all. And that is what junior high school mathematics is for.

We must first insist that the junior high school course in mathematics shall have a rich core of mathematical ideas. Some of the courses that have appeared have been amazingly thin. It is well known that at the end of the ninth year pupils in American high schools are in mathematics a year behind pupils of that grade in European secondary schools. The junior high school gives us a chance to shorten that gap. Junior high school administrators have not always been alive to the possibilities for improvement in scholarship. In discussing the aims of junior high schools Betz says: "Extremely significant, if not depressing, is the low rating given to the aim which stresses 'better scholar-

ship.''" Many courses in mathematics in junior high schools differ little from the conventional course given in the same grades in schools organized on the 8-4 plan.

There are certain well-defined tendencies that in recent years have greatly modified the subject matter and the teaching of mathematics in the high school, and especially in the junior high school. I shall discuss the influence of each of these tendencies on the materials and methods in arithmetic, algebra, and geometry in the seventh, eighth, and ninth grades.

1. Mathematics has been made more intuitive and less abstract in its beginnings.

We may find four stages in the growth of our ability to make mathematical abstractions.

(a) Going from groups of objects to the number of the group, from Σ to the number five.

(b) Going from particular numbers to general numbers, from 5 to a .

(c) Going from a to a particular function of a , from a to $\log a$, $a^2 + 4a + 7$, $\sin a$.

(d) Going from particular functions to general functions, from $\sin a$ to $f(a)$.

Good primary teaching has long since known the need for care in making the first step, and has developed materials and technique to put a firm basis of quantitative and numerical experience under elementary mathematical notions. Children see numbers and quantities of things, handle, count, and measure them and are led very slowly, with many returns and much practice in constructing images, in seeing things in their heads, from the objects to the ideas and symbols.

Contrast that with methods that have been in use in high school and college. There we have begun with the symbols, and I greatly fear often ended with them. We have been taking boys and girls who have been paddling around in a pool of particular numbers, 6, $14\frac{1}{2}$, 7.82, and have plunged them head-first into a sea of a^3 's, x^n 's, and $(a^8 - b^3)$'s. No wonder they sputter, choke, and in a large percent of the cases do not come up the third term. Many a time pupils have been put to proving all right angles equal before they knew the meaning either of angle or of prove. Has any one of us in college been allowed to make his own bridge from a few particular functions to the general theory?

That of course is not a true picture of present conditions in good schools. Instead of beginning with a mature and logically complete algebra, for example, an extended study of the fundamental operations with symbols that have little meaning, the pupils are coming into contact with algebraic ideas and processes as the race did, in finding answers to general questions and in expressing general relations between numbers. There is some sense in representing numbers by letters and in learning how to compute with them when you can set down once for all and in a most economical form the rule for finding the horsepower of an automobile and can use that formula in a case that arises in experience. High school teachers are devising materials and technique to help pupils in making this second step in mathematical generalization, to go from 5 to a .

The facts of geometry, both for the race and for the individual, have their roots deep in sense experience. Theory comes much later. Shall we begin the teaching of geometry with the geometry of the Egyptians or the geometry of the Greeks? We have made much progress in answering that question lately. That progress has been largely in the junior high school.

2. Some fruitful elementary notions from what has been called higher mathematics are presented in a concrete way and shown to be useful in solving problems. Such are the trigonometric ratios, logarithms, and the notion of a function. This passing down of powerful and useful notions will continue and will gradually modify, and I believe improve, our offerings in the high school. Why should not the high school pupil come into possession of some of the powerful modern ideas of algebra and geometry and the fundamentals of the calculus? By pushing some of the elements of algebra and geometry down into the seventh and eighth grades it will be possible to offer the beginnings of analytics and the calculus in the senior high school as the National Committee recommends. This attempt to present and to give meaning to a larger body of useful ideas which will later be enlarged and refined is one of the most important and useful of modern tendencies.

3. Along with the bringing in of these ideas of more advanced mathematics there goes the tendency to make high school mathematics more useful. The use of trigonometric ratios, logarithms, and notions of variation and function make possible the solution of problems that actually occur in practice.

4. This introduction of new materials has been made possible by the elimination of much material from the older courses. From arithmetic there has been eliminated obsolete topics and considerable problem material; from algebra complicated manipulations; and from geometry difficult and complicated proofs.

A striking thing about the mathematics courses for the junior high school is their plasticity, a hopeful feature that permits growth. There are many plans and all that I have observed are changing. No critic of mathematics teaching can yet say that junior high school mathematics is in a rut. It has not yet even worn a track fairly smooth. The plans differ as to the mathematical ideas to be presented, but more as to the order and method of the presentation. These plans all contain some arithmetic, some algebra, and some geometry.

Arithmetic.—The conventional course in arithmetic in the seventh and eighth grades was and is essentially a course in percentage and mensuration. It is singularly devoid of new mathematical ideas. The only mathematical ideas that have not already been taught in the first six grades are some geometrical definitions and rules with perhaps a glimpse of ratio, proportion and square root. The most of the time is used in applying numerical processes, already familiar, to scores of problems that differ not at all in the mathematical relations involved, only in vocabulary. Here is an example of what I mean. One of the most widely used seventh and eighth grade arithmetic texts of the last decade pointed with pride to the list of 52 occupations to which applications were made. What were the applications? One page was given to the "Pearl Button Industry." After a paragraph about the industry there were four or five problems using a pearl button vocabulary but involving only very simple mathematical relations of types that had already appeared over and over again. The treatment was too short to give any interesting or valuable information about the pearl button industry, there was no mathematical advance, and there could be scarcely any gain in ability to solve problems.

The changes going on generally in arithmetic have been accelerated in the junior high school. The enrichment recommended by the Committee of Fifteen more than thirty years ago has been made possible by the elimination of some outgrown topics but mainly by omitting quantities of problems of the type that I have just described.

I shall not outline in detail the arithmetic that is given in the junior high school. It is generally agreed that it shall contain a review for speed and accuracy of the fundamental operations with integers and common and decimal fractions; the elements of commercial arithmetic; numerical geometry or mensuration; and other applications to the affairs of everyday life. I believe many junior high schools offer little but arithmetic in the seventh and eighth grades.

The plans for the distribution of this material differ widely even in curriculums that offer some algebra and geometry in the seventh and eighth grades. Usually the simpler parts of percentage and mensuration are taught in the seventh grade, and the more difficult parts in the last half of the eighth. If the organization of the high school permitted, it would be much better to postpone the more difficult parts of commercial arithmetic until later when the pupils will probably have a background of experience that will enable them to understand it better. Subjects like stocks and bonds have no mathematical difficulties. Pupils have trouble with this subject because they do not understand the vocabulary and the business conditions.

Algebra.—The natural approach to algebra is through the formula. Everybody is keen to get hold of a labor-saving device, especially one to save thinking. If we were good advertisers the formula might be made as popular as an automatic dishwasher or a machine to grade testpapers. Who would not want to learn to use a machine that gives the solution of all problems of its kind at once, that saves learning, remembering, saying, and writing a long rule, and by easily mastered devices can be transformed into a new machine that solves other problems as difficult as the original? The formula $i = prt$ not only tells me how much my investment earns, but if put into reverse tells me how much I must have laid up to have a living income, or how long till I accumulate a given sum, or again what rate I should be assured of. This is a four speed formula, one forward and three in reverse. Have we made enough of the power, elegance, and economy of thought in algebraic symbolism?

By the use of formulas the pupils may solve many problems about distances, rates, and horsepowers that without formulas would be so involved in words as to be quite incomprehensible.

Arithmetic gives us particular answers to problems. Algebra

may enter as a natural result of attempts to get general answers. I believe every mind gets joy in the feeling of power that comes with making generalizations; not from memorizing generalizations of other people, but in actually making them.

Mathematics has two aspects. In one it faces applications in time and space and in the other the inner relations of mathematical truths. I am interested in the Pythagorean theorem because it helps me to find heights and distances. It also interests me in its relations to other geometric theorems. The first of these aspects comes first in the development of the individual. But the mathematical experience should not end there. Every educated person should see the "importance of mathematics as an instrument of natural conquests and social organization, and should be able to appreciate the value and significance of an ordered system of mathematical ideas." In opening up the domain of mathematics the junior high school should cultivate appreciation for both of these aspects.

Our teaching of algebra has had these glaring faults:

1. A sudden plunge into a highly abstract and formal set of ideas and symbols.
2. Too much attention to the mechanics.
3. Too short an exposure.

The older algebra had more definitions, principles, rules, and mechanical computation in the first 100 pages than is now in the whole introduction given the seventh and eighth grades and the first three months of the ninth. Algebra was then mainly a set of symbols and rules for combining them. Most of the time given to algebra in the ninth grade was given to performing the fundamental operations with integral and fractional expressions, factoring, and solving equations. More attention is now given to making algebraic symbols have a meaning. Algebra is becoming a way of answering questions that may be of interest, a labor-saving device, a tool to solve problems. The student now may not study so many rules and definitions, but he knows more, has a better start in mathematics. His time is more taken up with thinking mathematical relations, less with merely working with symbols.

Our practice has been to give one year of algebra in the ninth grade with no preparation and with no follow up. It seems to me absurd on the face of it to expect one year's exposure to the

material of the average ninth grade algebra text to give a mastery that is retained. Is it done in any other subject anywhere in the whole school system? If one year of calculus were required of everybody in college, would the results be any more satisfactory? I think not. The experience of other countries should mean something. Algebra is begun in the sixth grade in France and in the seventh in Germany and Austria, and runs through six or eight years. There is time to give abstractions meaning, and to fix ideas by use and to arrive at a general statement of them. That is the method of good teaching in arithmetic. It should be in algebra.

In junior high schools where there has really been a revision of the course in mathematics there is a beginning of algebra in the seventh grade, considerably more in the eighth, and the completion of what has been given as first year algebra in the ninth. The use of letters to represent numbers is begun with writing rules as formulas, $A = ab$, $p = rt$, $i = prt$, and in finding the answers to general questions: At 6 cents a pound what is the cost of 5 lb. of sugar? Of a lb.? Of $3x$ lb.? At c cents a pound what is the cost of 5 lb. of sugar? Of a lb.? Of $3x$ lb.? Since dividing the product by one of two factors gives the other, from $A = ab$ the pupils are taught how to solve for a and b . The meanings of the fundamental operations of algebra are learned by substituting in these formulas. At the end of the seventh year pupils should feel that it is rational to represent numbers by letters, should be able to compute with formulas, and know how to make and interpret simple graphs. Graphs are a connecting link between the algebra and the geometry of the year.

The algebra of the eighth year is an extension of that of the seventh. Formulas, graphs, and equations are used in solving problems. As there comes to be more use for adding, subtracting, multiplying and dividing with letters those topics are taught. There is an attempt to give much practice with simple operations. At the end of the eighth year the pupils should be able to solve simple equations, make, interpret, solve, and compute with formulas, make and interpret graphs, and perform the fundamental operations with simple algebraic expressions involving positive and negative numbers.

Ninth year mathematics is algebra in most schools and in most

junior high school texts. In schools that have had the preparation that I have recommended for the seventh and eighth grades more can be done and the mastery at the end of the ninth year should be better. A great limitation to progress in improving high school algebra it seems to me is that ninth grade texts designed for use in the junior high schools are in most cases made so that they may be sold to schools that have no algebra in the seventh and eighth grades, and as a result these books do not make the advance in algebra in the ninth grade that the previous training makes possible. The principal gain that results from the new course is a better mastery of the algebra that is taught. There should be gain in the amount of algebra taught.

Geometry.—Numerical geometry, or mensuration, has long been a part of the work of the seventh and eighth grades. In that we were taught definitions and rules for finding areas and volumes. The definitions were more or less formal and the rules were learned more or less mechanically according to the teaching. In the tenth grade we began demonstrative geometry. I began with trying to prove all right angles equal. You were brought up on a later vintage of texts and probably began with congruent triangles. Now demonstrative geometry has two huge initial difficulties, one of ideas and the other of method. Most pupils begin demonstrative geometry with very vague notions of the definitions of geometric figures and little knowledge of their properties. The elementary notions come from experience. "If you do not know what an angle is I can not tell you," said a wise old Greek. Most pupils do not know what an angle is and they can not use the idea in questions involving size, variation, comparison, proof. The form of argument of demonstrative geometry is new, concise, formal. It is a difficult tool to use at all, and requires much practice to be used with facility and accuracy. We often start the tenth grade pupil to applying this new method to a set of ideas that are vague, and as a last straw we start him to proving not the facts that may seem to him to need proof but those that are quite evident. No wonder there is confusion.

Here again there has been improvement recently. Texts in geometry for the tenth grade are now beginning with a chapter designed to illustrate geometric definitions, properties, relations, and the need for geometric proof. This if well taught may

greatly lessen the initial difficulties. The limitation of this procedure as I see it is that the introduction is too brief and appears to be only a curtain raiser for the big show. We need a longer introductory course, with definite objectives, and with enough time on the programme to realize them.

These are some of the ends to which such a course should contribute:

1. Useful information, including that given now in mensuration.
2. The use of the ruler, compasses, and protractor as tools for solving practical problems and in discovering and testing geometrical truths.
3. Appreciation of the value of geometry in industry and art.
4. Feeling the need for geometric proofs and making easy ones.

The excellent article by Betz on Junior High School Mathematics in the First Yearbook of the National Council of Teachers of Mathematics gives this statement of the case:

"Historically mathematics was called into existence because of the need for counting and measuring. This fact has given to mathematics a double foundation, namely arithmetic and geometry. It is apparent that we can not make or manufacture the simplest article without giving due attention to its form, its dimensions, and the proper relations of its parts. Nature and the manual arts are readily seen to be the two permanent sources of geometry. More especially, training in space intuition and plastic thinking is at the bottom of all forms of applied art. The geometric principles of equality, symmetry, congruence, and similarity are implanted in the very nature of things. The art of measurement permeates the fabric of modern civilization at every point. Then, too, intuitive geometry has a functional aspect through the unique training which it affords in the discovery and formulation of relationships. Finally, intuitive geometry is absolutely essential as a preparation for effective work in demonstrative geometry."

"When thus conceived, intuitive geometry serves to vitalize and humanize the whole course in elementary mathematics."

Junior high school courses usually give a half year to such a course in geometry. It includes work with ruler, compasses,

and protractor; mensuration; the discovery of many properties and relations of geometric figures; attention to the uses of geometry; and as a rule gives some practice in drawing conclusions from given facts. This work is usually given in the seventh year. It is probably the simplest and most easily mastered part of junior high school mathematics; but can be made also one of the most interesting, stimulating, and liberalizing parts.

Most of the texts in junior high school mathematics follow approximately the following plan. The seventh year's work contains about half a year of intuitive geometry of the type that I have been describing; a review of the fundamental processes of arithmetic with integers and common and decimal fractions; and some work with formulas and graphs. The eighth year's work contains the more difficult parts of commercial arithmetic and mensuration, and some algebra, including work with formulas, equations, graphs, and simple exercises with the fundamental operations with positive and, perhaps, negative numbers. The ninth year's work is algebra. Numerical trigonometry usually appears in the eighth or ninth year.

Concerning pupils from junior high schools who are to have separate classes in the tenth year, that is, who are not to be put into classes with pupils from schools with the eight-four plan, I wish to raise this question. Is this course of ninth year algebra the best course that we can offer these pupils to carry out the recommendation of the National Committee which would offer to the pupil the "most valuable mathematical training that he is capable of receiving in that year, with little reference to the courses which he may or may not take in succeeding years"? I do not believe this is the best course that may be planned for such a pupil. I believe that there is more education in some training in demonstrative geometry, and I would offer it in the last half of the ninth year. With a foundation of facts laid in the seventh and eighth years it is possible to give enough demonstrative geometry in a half year to be worth while. I would include in such a course only the simpler theorems that can be proved by direct methods, and I would see to it that the pupils make most of the proofs. I know that such a course can be made successful and that based on it a tenth grade class can complete the more difficult parts of plane geometry, solid geometry, and have time left for some further study of algebra.

May I give a bit of my own experience? In the training school at the Eastern Illinois State Teachers College we have been offering in the seventh and eighth grades for perhaps ten years such courses as I have recommended. In these two years the pupils got approximately one year of arithmetic, a half year of observational and intuitive geometry, and a half year of algebra. We then had the problem of planning the ninth year's work so that these pupils might make use of this extra training that had not been given to other ninth grade pupils who came into our high school from other grade schools. With this start in algebra we thought that the pupils from the eighth grade of the training school might in a half year complete ninth grade algebra as recommended by the Report of the National Committee. We believed that some training in demonstrative geometry, following the work in concrete geometry in the seventh grade, would be the best training we could offer either the pupil who intended to continue mathematics or the one who took no more. So we planned a half year's work in algebra for the first half of the ninth year and demonstrative geometry for the second half. The work in demonstrative geometry was planned with the assumption that a considerable body of facts had been learned in the seventh grade. We did not then take time for the usual introduction, but began immediately to give proofs. Since the pupils knew most of the facts that they were asked to prove their attention could be given to the method of argument. We purposely chose theorems that could be proved by direct methods, and attempted to grade the difficulties so that the pupils might make most of the proofs themselves. In many cases only suggestions of lines of proof were given. We found that we could give in the half year the greater part of the fundamental theorems listed by the Report of the National Committee.

Year before last I taught the tenth grade class that had taken the work of the seventh, eighth, and ninth grades that I have just outlined. Our school year of 36 weeks is divided into three terms of 12 weeks each. In the first two terms the class completed the theorems of plane geometry that are ordinarily given in the tenth year, and also the work in solid geometry from a standard text. I thought that the work was done at least as well as in high school classes that have plane geometry through-

out the tenth year and a half year for solid geometry in the eleventh. This class in the third term of the tenth year did work in algebra under another teacher. I think that they probably did rather less than is done in the third year of high school algebra. Our tenth year class last year was not so good a class and had a bad experience in the ninth year through the sickness and resignation of the teacher who began the ninth year's work. At the end of the tenth year they completed plane geometry and considerable of the work of the third half year of algebra. This year's class promises to do better than last year's. From this experience it is fair to conclude that by this programme the pupils at the end of the tenth year are at least a half year ahead of those who do algebra in the ninth and geometry in the tenth. I think that because of the more careful and extended introduction to both algebra and to the continued exposure over a longer period the mastery is better.

There has been much talk in the last ten years about the function concept being the core of the mathematics course in the high school. We are still a long way from such an organization of materials. We have made some progress in that direction, and the attempts to introduce the notion of function and variation have done a good deal to improve problem material.

The course that I have suggested is the one that I think represents a fair ideal. I can not say that it is near the practice of most junior high schools. In fact the outlines of most courses that I have seen indicate that they may not differ much from the course of arithmetic in the seventh and eighth grades and algebra in the ninth that most of us were brought up on. There is great need that teachers who are interested enough in their problems to attend such conferences as this should use their influence in giving junior high school pupils better courses in mathematics.

NEW BOOKS

Modern Methods of Teaching Arithmetic. By RALPH S. NEWCOMB. Boston: Houghton Mifflin Co., 1926. Pp. xv + 345. \$2.00.

In this book Mr. Newcomb has attempted to meet the demands of the modern conception of the purposes of instruction in arithmetic and to adapt methods and devices in accordance with the general pedagogical and psychological principles accepted to-day. The field covered includes not only the customary topics such as fundamental operations, common and decimal fractions, percentage, drill, and problem solving, but also a study of tables, statistics, graphs and algebra, and geometry suitable for the elementary school as well as a short discussion of the curriculum carried further in a chapter on Socialization and Correlation of Arithmetic and surveys briefly the available standard tests. Constant reference has been made to the field of experimental psychology for the rejection or substantiation of methods discussed. However, throughout the entire book, the reviewer finds but two references to publications later than 1924 and suggests that some use might well have been made of some of the valuable contributions made in the last two years. The book is set up after the manner of a textbook with discussions under topical headings and questions at the end of each chapter, so is well adapted for classroom use in teachers' colleges and normal schools.

Horace Mann Supplementary Arithmetic. By HILLEGAS, PEABODY, BAKER. Philadelphia: J. B. Lippincott Co., 1925. Pp. 156.

Teaching Number Fundamentals. By MILO B. HILLEGAS. Philadelphia: J. B. Lippincott Co., 1925. Pp. 98.

Teaching Number Fundamentals is a teachers' manual which accompanies the *Horace Mann Supplementary Arithmetic* to give directions for the use of the drill book and to explain the principles upon which it was developed. The manual is divided

into three parts, the first of which includes a brief discussion of the need for drill material and the salient features of this particular drill material; the second part is given over entirely to directions for the teachers' guidance in using the material; and the third part contains an analysis of the exercises in the drill book.

Horace Mann Supplementary Arithmetic (diagnostic and corrective) provides practice material for addition, subtraction, multiplication, short and long division. Each process is divided into three parts. The first part includes basic facts, the second and third parts contain steps graded in difficulty. For each step in the sequence two equivalent sets of examples are provided. If the child fails on the first set, he takes the second. It provides good practice, exceedingly well graded, but can scarcely be considered diagnostic in as much as it tells only what the child cannot do and not why he is unable to do it.

Social Arithmetic—Book One. By FRANK M. McMURRAY and C. BEVERLY BENSON. New York: The Macmillan Company. Pp. v + 345.

The authors have written this book with two apparent principles in view: (1) that it was not for children but to children, (2) that it should utilize such available information as could possibly be adapted to the field. They have no concern about the teacher as is evidenced by the fact that there is neither an editor's nor an author's preface, the only introduction being a few pages addressed "to the boys and girls who have this book." It sets up situations involving the fundamental processes through long division and common fractions and provides some material which is pure drill. The chapter titles are significant enough to merit mention of a few of them here: What Tom Learned about Buying Groceries, Important Facts about Numbers on the Farm, Where People Live and Work in Large Cities. The problems are grouped together in paragraphs so as to give the book the appearance of any book of contact.

Despite the remarkable ingenuity displayed by the authors in finding and using actual life situations one cannot but feel that they have perhaps strained the point occasionally. The reviewer thinks it might be an interesting method of teaching arithmetic, if not an economical one.

An Arithmetic for Teachers. By WILLIAM F. ROANTREE and MARY S. TAYLOR. New York: The Macmillan Company, 1925. Pp. xiii + 621.

In *An Arithmetic for Teachers* the authors have combined material which will provide for the teachers' knowledge of the subjects with discussions of the methodology of teaching arithmetic. They have been guided by a belief that a teacher should have (1) The explanation or definition of every term occurring in the subject of arithmetic that he might be called upon to explain, (2) a proof for every rule and principle occurring in the subject of arithmetic, (3) an explanation of the steps in the performance of any of the operations of arithmetic, a discussion of their logical sequence, and a good form for the written work in the operations, (4) a full explanation of how to solve problems arithmetically, a brief explanation of various other methods of solution, and an analysis of what constitutes good problem material, (5) a few historical notes on the various topics taught in arithmetic, (6) approved methods of teaching the different topics, (7) an analysis of different types of lessons to be taught and of the method which is best adapted to each. They have devoted more space to the problem of business and economics than is used, putting considerable emphasis on interest, commercial paper, insurance, stocks, bonds, building loans and taxation. It will not only be useful for classes in the teaching of arithmetic but a valuable addition to the library of every teacher in service who comes in contact with arithmetic teaching.

What Arithmetic Shall We Teach. By GUY M. WILSON. Boston: Houghton Mifflin Company. Pp. viii + 149. \$1.20.

Mr. Wilson would approach the problem of curriculum construction from an analysis of life's needs and in *What Arithmetic Shall We Teach* he summarizes the conclusions at which he has arrived from his own studies and a survey of the related investigations of others. His own recent studies bear out with only slight variations the conclusions drawn in his study *A Survey of the Social and Business Usage of Arithmetic*, 1918. The writer studied the topics of arithmetic in accordance with their use in various vocational fields, interpreting the results by applying the criteria of frequency of use, cruciality, adaptability, and relative values. He couches his conclusions in the two last

chapters in which he summarizes the processes and the degree of difficulty for the grades, and sets up the new course of study. His data are undoubtedly valid, his conclusions apparently sound and the book a valuable contribution to the science of curriculum making.

JOSEPHINE HALEY

THE LINCOLN SCHOOL

General High School Mathematics, Books I and II. By DAVID EUGENE SMITH, JOHN A. FOBERG, and WILLIAM D. REEVE. Ginn and Company.

"Why teach all this factoring and purposeless algebraic manipulation during the first semester when it is so fatal?" Mathematics teachers often ask themselves this question, and should therefore welcome with joy the appearance of a new textbook which has the courage to incorporate a more vital organization of material for the senior high school.

General High School Mathematics, Books I and II, are designed for use in the first two years of a four-year high school course. Book one is essentially a vitalized algebra, and book two a vitalized, modernized geometry.

Book one is designed to "give an appreciation of the nature and uses of algebra, an acquaintance with the facts of intuitive geometry, and an insight into the simple uses of trigonometry." Book two is fundamentally demonstrative geometry including both plane and solid geometry units and also some trigonometry.

Both books contain ample material to provide room for selection by the teacher, and can therefore be used in any ninth and tenth grades. A ninth-grade class having had only arithmetic in the seventh and eighth grades should take the intuitive geometry work from the beginning of Book I. A ninth-grade class having had the newer type of work in the seventh and eighth grades could well skip most of the first four chapters.

The use of letters is skillfully developed through the formulas of intuitive geometry, and the use and solution of equations follows a rationally developed unit on directed numbers. Factoring is introduced late in the course, and largely as a means of solving quadratic equations. Other customary algebra units follow, and then a chapter on numerical trigonometry. The last chapter provides a general review drill which should meet the approval of psychology.

In book two the pupil is made to feel the need for proof and a genuine effort is made to develop in the pupil the ability to reason with and about geometrical magnitudes. The outstanding feature of this book is the teachableness of its contents and the practical and interesting nature of its exercises.

Two units of work with logarithms and trigonometry, including the solution of oblique triangles, are introduced before the chapters on solid geometry. Evidently the purpose here is to provide a possible enrichment of the course for some who may be able to do the work but who might not go on with mathematics.

The outstanding features of the books are their teachableness and their adequate recognition of the psychology of drill. The books are a methods course as well as textbook material. The authors have done a very skillful piece of work in the presentation of a two-year course in vitalized mathematics for the senior high school.

Arithmetic for Teacher-Training Classes. By E. H. TAYLOR.
Henry Holt & Co., New York. 1926.

Mathematics courses in the teaching of arithmetic in Normal Schools have been so concerned with methods and devices that students who are later to be teachers have found themselves, and have been found, with very inadequate scholarship in the science itself. One suggestion of a way to overcome this deficiency in teacher-training classes is to have two courses, one a sound subject matter course in the fundamental principles of arithmetic, followed by a professional course in which the psychology and method of learning and teaching are emphasized. The other suggestion is that of combining in one course the study of both subject matter and of method of teaching.

This new text by Professor Taylor is intended to prepare the student by giving methods of procedure, as how to teach the multiplication of decimals; by the analysis and gradation of difficulties, as in teaching long division; by developing topics for the student's own understanding in ways that he in turn may use in teaching his own pupils; by questions on teaching that are given at the ends of chapters. These questions are planned to familiarize the student with the best reference books on the teaching of arithmetic, and with the results of recent research which has aimed to discover specific difficulties that children

meet, methods of overcoming them, and the uses of arithmetic in child and adult life.

The author has emphasized the content of arithmetic, the mathematics of arithmetic we might say, more than he has the psychology and pedagogy of arithmetic.

NEWS NOTES

THE next meeting of the Association of Teachers of Mathematics in the Middle States and Maryland will be held in Milbank Chapel, Teachers College, Columbia University, May 7.

The forenoon program (9 to 12) includes:

1. Conference on organization and plans.
2. Curriculum Problems, Professor W. D. Reeve.
3. Functions in General and the Function (x) in Particular, Professor Walter B. Carver.
4. The Determination of Teaching Problems in Algebra, Dean J. H. Minnick.

The afternoon program (2 to 4) includes:

5. Improving Instruction in Geometry, Professor John R. Clark.
6. Pre-College Calculus in the Horace Mann High School, Miss Vevia Blair.

The officers of the Association are: President, Professor Wilfred H. Sherk, University of Buffalo; Vice-President, Dr. E. E. Rice, The Lawrenceville School, Lawrenceville, New Jersey; Secretary, Professor W. D. Reeve, Teachers College, Columbia University; Treasurer, Miss Elsie O. Bull, State Normal School, West Chester, Pa.

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